Società Nazionale di Scienze, Lettere ed Arti, in Napoli
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Rendiconto, Anno 2014

Beyond Feynman's troubles in Electromagnetics

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(Adunanza del 4 aprile, 2014)

Key words: Electromagnetic fields, differential forms, balance laws, flux rules, Lorentz force law.

Abstract - New basic principles of electric and magnetic induction are formulated as balance laws referring to material paths undergoing piecewise discontinuous motions. The resulting differential rules are able to overcome difficulties of standard formulations and to provide a direct interpretation to experimental evidence involving induction phenomena whose description in terms of flux rules is troublesome or unfeasible. FARADAY-HENRY-NEUMANN-FELICI and AMPÈRE-MAXWELL flux laws of electromagnetic induction are recovered in the special case of closed material circuits undergoing regular motions. For an electric charge translating into a uniform magnetic field due to a spatially time-invariant magnetic potential, a term analogous to LORENTZ force is found to provide the induced electric field, but with a correction factor of one-half.

Riassunto - Principi di bilancio che governano i fenomeni di induzione elettrica e magnetica sono formulati con riferimento a linee materiali in moto, anche discontinuo a pezzi. Le conseguenti regole differenziali sono suscettibili di interpretare in modo diretto le evidenze sperimentali anche quando le formulazioni standard risultano di problematica o impossibile applicazione. Le leggi di flusso, come enunciate in letteratura, sono ritrovate come caso particolare per circuiti materiali chiusi in moto non discontinuo. Per cariche elettriche in moto traslatorio in un campo magnetico uniforme dovuto ad un potenziale magnetico costante nel tempo, si ritrova la cosiddetta legge di forza di LORENTZ come espressione del campo elettrico, ma corretta da un fattore un mezzo.

1 PROLEGOMENA

History of electromagnetism is really fascinating. Starting from the many brilliant experimental discoveries and interpretations by ROMAGNOSI, ØRSTED,

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BIOT, SAVART, AMPÈRE in 1800-1820, and ZANTEDESCHI-HENRY-FARADAY-NEUMANN-LENZ-WEBER-FELICI in between 1829 and 1855, early beautiful theoretical abstractions soon led to the formulation of GAUSS-FARADAY's and AMPÈRE-MAXWELL laws of electromagnetic induction.

It is however to be said that, in looking at modern treatments of the fundamentals of electromagnetism, a careful reader would certainly agree with R.P. FEYNMAN in being disappointed by the contamination of the synthetic and powerful original principles with *ad hoc* additional rules aimed at interpreting the role of body motion in induction phenomena (Feynman et al., 1964, II.17-1).

It seems that troubles became to appear in the scientific literature as far as the analysis conceived by Clerk-Maxwell (1861) was subjected to simplifications proposed, around the end of the nineteenth century, in (Heaviside, 1892; Hertz, 1892; Lorentz, 1895, 1899, 1903, 1904).

These modifications were performed with an insufficient attention to the original theoretical framework set up in (Clerk-Maxwell, 1861, 1865, 1873) and in (Helmholtz, 1870, 1873, 1874, 1892).

The main negative feature consisted in the way motions of material particles were taken into account. In fact they were either completely ignored, as feasible only in getting the wave equations *in vacuo*, or adjusted by adding *ad hoc* terms in the case of simple relative translations, with the consequence that invariance under frame transformations was lost.

According to a prevailing opinion (Darrigol, 2000), the treatments by HEAVISIDE and HERTZ improved MAXWELL analysis by taking care of particle motion. A reading of the original papers (Clerk-Maxwell, 1861, 1865) and (J.J. Thomson, 1893) is however sufficient to completely disprove such statements and to put into evidence the power of the original formulation.

Another peculiar occurrence was the appearance of vector calculus which, introduced by GIBBS in 1888, was soon adopted to simplify the previous analysis based on the quaternion algebra, and, published in extended form in (Gibbs, 1929), rapidly became the standard formalism in physics and engineering of the twentieth century.

Unfortunately, with the vector symbolism, physical entities were deprived of their proper geometrical nature and flattened on a common algebraic platform.

The advent of the relativistic era was of no advantage in revising the flaws introduced in the theory of electromagnetics by the cited simplifications.

On the contrary, under the influence of the 1905 paper (Einsten, 1905), the attempts of recovering frame invariance of MAXWELL-HERTZ laws of induction, led to sustain that EUCLID frame transformations ought to be substituted by LORENTZ transformations.

This mathematically unfounded conclusion was taken for granted in the subsequent literature, therefore becoming a main stoppage for an improvement of the theory, due also to misstatements concerning the transformation of electromagnetic fields under LORENTZ transformations (Romano G., 2013).

To give an answer to the problem of frame invariance, it would have been sufficient to revert to MAXWELL original treatment, as improved by J.J. THOMSON, and to elaborate on it with the invariant formalism of differential forms (Romano G., 2013).

The new theory exposed below is formulated to fulfil two basic guidelines. An interpretation as direct as possible of experimental evidence, and a geometrically correct mathematical modelling. The theory develops entirely in the classical framework and consists in the adoption of electric and magnetic induction laws formulated as balance principles pertaining to arbitrary curvilinear paths undergoing piecewise regular motions.

The power of the theory in providing clear interpretations to experimental evidence is shown by several applications in Sect. 6.

Flux rules are recovered for closed paths undergoing regular motions, but, at difference with the current usage in literature, the LORENTZ *force rule* is not assumed to be a basic law of the theory, being reduced, after a correction by a factor one-half, to a special expression detected by special observers investigating special situations, as shown in Sect. 6.1.

2 DIFFERENTIAL FORMS

The theory of integration over compact manifolds, whether orientable or not, is treated in (Abraham et al., 2002) and resumed in (Romano G., 2007). We recall here only some basic issues, for future reference.

Definition 2.1 (Multi-covectors). *A k-covector ω^k in a n-dimensional linear space V ($k \leq n$) is an alternating k-linear real valued map on the tangent manifold TV.*

This means that the exchange of two argument results in the sign change the scalar value and we write $\omega^k \in \mathrm{ALT}^k(TV)$.

A non-null n-covector on V vanishes if and only if its arguments are linearly dependent and all n-covector are proportional one another.

A non-null n-covector ω^n provides the natural way of computing an n-volume of an n-parallelepiped.

Definition 2.2 (Differential exterior forms). *A* differential exterior form *of order k (or simply a k-*form*) is a smooth field of k-covectors defined on a n-manifold.*

Integral laws in Mathematical Physics are naturally formulated in terms of exterior forms because integrals over (inner) orientable manifold of dimension k are in fact evaluations of global k-volumes.

Therefore there is no surprise that the electromagnets theory, which is governed by integral balance laws, is most conveniently developed in terms of exterior forms representing electric charges and electric and magnetic fields.

Noteworthy contributions, with applications of differential geometric notions to theoretical and computational aspects of electromagnetism were provided by Deschamps (1970, 1981), Bossavit (1991, 2004, 2005) and Tonti (1995, 2002).

An important issue concerns orientations of manifolds (Schouten, 1951; Abraham et al., 2002).

We recall here just essential ideas. In a n-manifold a continuous volume n-form defines an (*inner*) orientation. The outer orientation is defined at each point by orienting a linear complement of the tangent space to the manifold.

For an orientable 2D surface the *inner orientation* on the surface is either clockwise or counter clockwise while the *outer orientation* through the surface is either from the negative face to the positive one or vice versa.

Similarly, for 1D paths an (*inner*) orientation is a direction of walk along it, while an *outer orientation* is a choice of turning around it.

For 3D manifolds an (*inner*) orientation can be either left-handed or right handed while an *outer* orientation is a source or a sink. The converse for a point (a 0D manifold).

The choice of an *orientation* in an orientable container manifold defines on its submanifolds an *outer* orientation associated with a given *inner* orientation. An inner (outer) orientation on a manifold induces an inner (outer) orientation on its boundary.

Integration of (*inner*) forms on an orientable manifold involves the choice of an inner orientation and the integral will change sign by changing the inner orientation. Integration on *outer* oriented manifolds involves *outer* forms that are defined by the property that they change sign on changing the orientation in spatial slices.

Chains of manifolds are generated by formal linear combination by integers with positive or negative coefficients depending on wheter the orientations induced on common boundaries are compatible or not.

Adoption of the mathematical language provided by the theory of differentiable forms is most compelling when dealing with investigations about transformations induced by diffeomorphisms, such as space-time motions or changes of reference frame.

The tangent map $T\zeta : T\Omega \mapsto T\mathbf{M}$ associated with an injective smooth map $\zeta : \Omega \mapsto \mathbf{M}$, transforms a tangent vector $\mathbf{a} \in T\Omega$ into a pushed forward vector

$$\zeta{\uparrow}\mathbf{a} = T\zeta \cdot \mathbf{a} \circ \zeta^{-1} \in T\mathbf{M} \tag{1}$$

tangent at the corresponding point on $\zeta(\Omega) \subset \mathbf{M}$. Here and in the sequel a dot $\cdot$ means fiberwise linear dependence on subsequent arguments.

The pull-back of a differential form ω on $T(\zeta(\Omega))$ by the map ζ is the differential form $\zeta{\downarrow}\omega$ on $T\Omega$ defined (assuming for simplicity a one-form) by

$$(\zeta{\downarrow}\omega)(\mathbf{a}) = \omega(\zeta{\uparrow}\mathbf{a}), \quad \forall \mathbf{a} \in T\Omega. \tag{2}$$

A physical field on a manifold $\mathbf{M}$, represented by a a differential form ω on $T\mathbf{M}$, is *invariant* under a smooth invertible transformation (automorphism) $\chi : \mathbf{M} \mapsto \mathbf{M}$ if

$$\omega = \chi{\downarrow}\omega. \tag{3}$$

A fundamental role is played by the notion of exterior derivative of a k-form on a manifold Ω, which extends to multidimensional manifolds the fundamental theorem of integral calculus of real functions.

Definitions and principal properties concerning exterior derivatives, recalled below, will be referred to in the development of the new theory of electromagnetic induction.

Definition 2.3 (Exterior derivative and Stokes' formula). *In a compact manifold Ω od dimension $(k+1)$, with a k-dimensional boundary $\partial\Omega$, the exterior derivative operates on a k-form $\omega \in \mathrm{C}^1(\Omega; \mathrm{ALT}^k(T\Omega))$ to give the $(k+1)$-form $d\omega \in \mathrm{C}^1(\Omega; \mathrm{ALT}^{(k+1)}(T\Omega))$ fulfilling* STOKES' *formula*

$$\int_\Omega d\omega = \oint_{\partial\Omega} \mathbf{i}_\partial{\downarrow}\omega. \tag{4}$$

where $\mathbf{i}_\partial \in \mathrm{C}^1(\partial\Omega; \Omega)$ is the embedding of the boundary manifold $\partial\Omega$ in the manifold Ω. The pull-back $\mathbf{i}_\partial{\downarrow}$ by the embedding is needed to transform exterior forms on $T\Omega$ to exterior forms on $T\partial\Omega$. For the sake of notational simplicity, it is often abusively omitted in STOKES' *formula.*

A manifold Ω with null boundary ($\partial\Omega = \mathbf{0}$) is said to be *closed*. A boundary $\partial\Omega$ is a closed manifold, i.e. $\partial\partial\,\Omega = \mathbf{0}$ for any manifold Ω.

From STOKES formula it follows that $dd\,\omega = 0$ for any form ω.

Analogously a differential form $\omega \in \mathrm{C}^1(\Omega; \mathrm{ALT}^k(T\Omega))$ is closed if $d\omega = \mathbf{o}$ and is exact if $\omega = d\alpha$ for some form $\alpha \in \mathrm{C}^1(\Omega; \mathrm{ALT}^{(k-1)}(T\Omega))$.

On manifolds that are contractable to a point every closed form is exact, a result known as POINCARÉ Lemma (Abraham et al., 2002; Romano G., 2007).

3 KINEMATICS

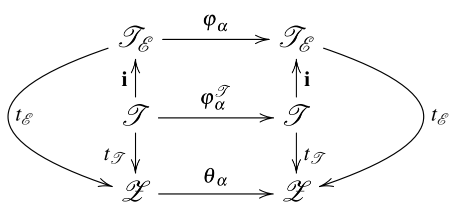
In the theory of electromagnetic induction the container manifold is the 4D manifold $\mathscr{E}$ of physical events.

An observer performs a partition of $\mathscr{E}$ into disjoint *spatial slices* $\mathscr{E}(t)$ each associated with a time instant $t \in \mathscr{Z}$ (*Zeit* is *Time* in German) by means of a time-projection $t_{\mathscr{E}} : \mathscr{E} \mapsto \mathscr{Z}$. The spatial slices $\mathscr{E}(t)$ are level sets of the *time-projection* so that $\langle dt_{\mathscr{E}}, \mathbf{d} \rangle = 0$ for any spatial vector $\mathbf{d} \in T\mathscr{E}(t)$.

A congruence of time-lines provides a one-to-one correspondence between spatial slices. The time parametrisation of time lines generates a vector field of tangent 4-vectors $\mathbf{Z} \in T\mathscr{E}$ such that $\langle dt_{\mathscr{E}}, \mathbf{Z} \rangle = 1$.

A smooth map $\mathbf{i} : \mathscr{T} \mapsto \mathscr{E}$ between a kD manifold $\mathscr{T}$, with $k \leq 3$, is an *immersion* if the tangent map $T\mathbf{i} : T\mathscr{T} \mapsto T\mathscr{E}$ is pointwise nonsingular. If moreover the co-restricted map $\mathbf{i} : \mathscr{T} \mapsto \mathscr{T}_{\mathscr{E}} = \mathbf{i}(\mathscr{T})$ is a diffeomorphism, the map is called an *embedding* and $\mathscr{T}_{\mathscr{E}}$ is a submanifold of $\mathscr{E}$.

The *space-time motion* of a body is detected by an observer as a one-parameter ($\alpha \in \mathscr{Z}$ time lapse) family of smooth transformations (automorphisms) $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}_{\mathscr{E}} ; \mathscr{T}_{\mathscr{E}})$, over the *trajectory* submanifold $\mathscr{T}_{\mathscr{E}}$, fulfilling the commutative diagram



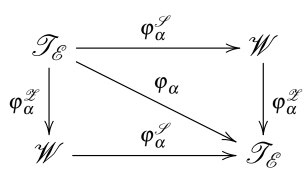
$$\begin{aligned} \varphi_\alpha \circ \mathbf{i} &= \mathbf{i} \circ \varphi_\alpha^{\mathscr{T}} \,, \\ t_{\mathscr{T}} \circ \varphi_\alpha^{\mathscr{T}} &= \theta_\alpha \circ t_{\mathscr{T}} \\ t_{\mathscr{E}} \circ \varphi_\alpha &= \theta_\alpha \circ t_{\mathscr{E}} \,. \end{aligned} \tag{5}$$

Here $\theta_\alpha : \mathscr{Z} \mapsto \mathscr{Z}$ is the time-translation defined by $\theta_\alpha(t) = t + \alpha$, so that Eq. $(5)_3$ means that the motion preserves simultaneity of events.

The motion *four-velocity* is $\mathbf{V} = \partial_{\alpha=0}\, \varphi_\alpha$ with $\langle dt_{\mathscr{E}}, \mathbf{V} \rangle = 1$.

The motion $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}_{\mathscr{E}} ; \mathscr{T}_{\mathscr{E}})$ draws on each spatial-slice $\mathscr{E}(t)$ a wake $\mathscr{W}(t)$ of events that are intersected by time-lines passing through the embedded trajectory. The union of all wakes is denoted by $\mathscr{W}$.

The space-time motion along the embedded trajectory are split by an observer into a *spatial motion* and a *time motion* fulfilling the commutative diagram



$$\iff \quad \varphi_\alpha = \varphi_\alpha^{\mathscr{S}} \circ \varphi_\alpha^{\mathscr{Z}} = \varphi_\alpha^{\mathscr{Z}} \circ \varphi_\alpha^{\mathscr{S}} \,, \tag{6}$$

The space-time velocity $\mathbf{V}$ is decomposed into the sum of *spatial* and *time* components $\mathbf{V} = \mathbf{Z} + \mathbf{v}$ with $\langle dt_{\mathscr{E}}, \mathbf{v} \rangle = 0$ and

$$\mathbf{v} := \partial_{\alpha=0}\, \varphi_{\alpha}^{\mathscr{S}} \,, \qquad \mathbf{Z} := \partial_{\alpha=0}\, \varphi_{\alpha}^{\mathscr{Z}} \,. \tag{7}$$

A form ω^k on $\mathscr{T}_{\mathscr{E}}$ is *spatial* if it vanishes when any of its k argument vectors, which are tangent to $\mathscr{E}$, is parallel to $\mathbf{Z}$.

The integral of the pull-back of a k-form ω^k over the placement $\varphi_{\alpha}(\Omega)$ is equal to the integral of the pull-back form over the placement Ω

$$\int_{\varphi_{\alpha}(\Omega)} \omega^k = \int_{\Omega} \varphi_{\alpha}\!\downarrow\!\omega^k \,. \tag{8}$$

Taking the time-derivative of Eq. (8) gives the LEE-REYNOLDS formula

$$\partial_{\alpha=0} \int_{\varphi_{\alpha}(\Omega)} \omega^k = \int_{\Omega} \partial_{\alpha=0}\, (\varphi_{\alpha}\!\downarrow\!\omega^k) = \int_{\Omega} \mathscr{L}_{\mathbf{V}}\, \omega^k \,. \tag{9}$$

The exterior derivative of differential forms commute with the pull-back by the motion

$$d \circ \varphi_{\alpha}\!\downarrow = \varphi_{\alpha}\!\downarrow \circ\, d \,, \tag{10}$$

a result following from STOKES and integral transformation formulae

$$\begin{aligned} \int_{\Omega} d(\varphi_{\alpha}\!\downarrow\!\omega^k) &= \oint_{\partial\Omega} \varphi_{\alpha}\!\downarrow\!\omega^k = \oint_{\varphi_{\alpha}(\partial\Omega)} \omega^k \\ &= \oint_{\partial\varphi_{\alpha}(\Omega)} \omega^k = \int_{\varphi_{\alpha}(\Omega)} d\omega^k = \int_{\Omega} \varphi_{\alpha}\!\downarrow\!(d\omega^k) \,. \end{aligned}$$

Then also

$$\mathscr{L}_{\mathbf{V}}\, (d\omega^k) = d\, (\mathscr{L}_{\mathbf{V}}\, \omega^k) \,. \tag{11}$$

More in general, a smooth automorphic transformation $\zeta_{\mathscr{E}} : \mathscr{E} \mapsto \mathscr{E}$ (a frame transformation) induces a correspondence $\zeta : \mathscr{T} \mapsto \mathscr{T}_{\zeta}$ between trajectories $\mathscr{T}$ and $\mathscr{T}_{\zeta}$, according to the commutative diagram

$$\begin{array}{ccc} \mathscr{E} & \xrightarrow{\ \zeta_{\mathscr{E}}\ } & \mathscr{E} \\ {\scriptstyle \mathbf{i}} \uparrow & & \uparrow {\scriptstyle \mathbf{i}_{\zeta}} \\ \mathscr{T} & \xrightarrow{\ \ \zeta\ \ } & \mathscr{T}_{\zeta} \end{array} \quad \Longleftrightarrow \quad \zeta_{\mathscr{E}} \circ \mathbf{i} = \mathbf{i}_{\zeta} \circ \zeta \,, \tag{12}$$

and we have the commutation rule

$$d \circ \zeta\!\downarrow = \zeta\!\downarrow \circ\, d_{\zeta} \,, \tag{13}$$

where d and d_ζ are the exterior derivatives acting respectively on the trajectory $\mathscr{T}$ and on the transformed trajectory $\mathscr{T}_\zeta$.

Accordingly, the motions detected in the two frames are related by the commutative diagram

$$\begin{array}{ccc} \mathscr{T}_\zeta & \xrightarrow{(\varphi_\alpha^{\mathscr{T}})_\zeta} & \mathscr{T}_\zeta \\ \zeta\uparrow & & \uparrow\zeta \\ \mathscr{T} & \xrightarrow{\varphi_\alpha^{\mathscr{T}}} & \mathscr{T} \end{array} \quad \Longleftrightarrow \quad (\varphi_\alpha^{\mathscr{T}})_\zeta \circ \zeta = \zeta \circ \varphi_\alpha^{\mathscr{T}}\,, \tag{14}$$

with space-time *four-velocities* related by $\mathbf{V}_{\zeta_{\mathscr{E}}} = \zeta_{\mathscr{E}}\!\uparrow\!\mathbf{V}$.

The LIE-*derivative*, along the space-time motion $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}_{\mathscr{E}}\,;\mathscr{T}_{\mathscr{E}})$, of a k-form ω^k on the embedded trajectory $\mathscr{T}_{\mathscr{E}}$, is the time-derivative $\mathscr{L}_{\mathbf{V}}\,\omega^k$ of the pull-back form which is pointwise a time-dependent multicovector in the relevant tangent linear space

$$\mathscr{L}_{\mathbf{V}}\,\omega^k := \partial_{\alpha=0}\,(\varphi_\alpha\!\downarrow\!\omega^k)\,. \tag{15}$$

LIE-*derivative* transform in a natural way by the effect of diffeomorphic transformations $\zeta_{\mathscr{E}} \in \mathrm{C}^1(\mathscr{E}\,;\mathscr{E})$

$$\zeta_{\mathscr{E}}\!\uparrow\!(\mathscr{L}_{\mathbf{V}}\,\omega^k) = \mathscr{L}_{(\zeta_{\mathscr{E}}\uparrow\mathbf{V})}\,(\zeta_{\mathscr{E}}\!\uparrow\!\omega^k) \tag{16}$$

Under the assumption that for sufficiently small time lapse the spatial and time motions do not bring outside the trajectory submanifold, the split (6) and LEIBNIZ rule lead to the additive decomposition

$$\begin{aligned} \mathscr{L}_{\mathbf{V}}\,\omega^k &:= \partial_{\alpha=0}\,(\varphi_\alpha\!\downarrow\!\omega^k) = \partial_{\alpha=0}\,(\varphi_\alpha^{\mathscr{S}} \circ \varphi_\alpha^{\mathscr{Z}})\!\downarrow\!\omega^k \\ &= \partial_{\alpha=0}\,\varphi_\alpha^{\mathscr{S}}\!\downarrow\!\omega^k + \partial_{\alpha=0}\,\varphi_\alpha^{\mathscr{Z}}\!\downarrow\!\omega^k = \mathscr{L}_{\mathbf{Z}}\,\omega^k + \mathscr{L}_{\mathbf{v}}\,\omega^k\,. \end{aligned} \tag{17}$$

Intersections of embedded trajectory $\mathscr{T}_{\mathscr{E}}$ with spatial slices, are assumed to be compact kD submanifolds Ω called *placements* of the body.

3.1 HOMOTOPY FORMULA

The next result provides a generalisation of a well-known formula introduced by HELMHOLTZ for the evaluation of the time-rate of variation of the flux of a vector field, see Sect. 3.2.

The formula was later reformulated in modern geometrical terms by H. CARTAN to deal with LIE derivatives of differential forms of any order along the flow associated to vector fields on a manifold.

The version we present here is further extended to include the representation of the LIE derivative along the space-time motion, of a form ω^k defined on the $(k+1)$D trajectory submanifold $\mathscr{T}_{\mathscr{E}}$ of the 4D event manifold $\mathscr{E}$ $(k \leq 3)$.

As will be shown, this extension is essential to get the material homotopy formula, stated below in Prop. 3.1 and in Cor. 3.1, and proven in (Romano G., 2007).

Proposition 3.1 (Homotopy formula). *The* LIE*-derivative* $\mathscr{L}_{\mathbf{V}}\,\omega^k$ *of a space-time form* ω^k *along the motion, is expressed in terms of exterior derivatives by the homotopy formula*

$$\mathscr{L}_{\mathbf{V}}\,\omega^k = (d_{\mathscr{T}}\,\omega^k)\cdot\mathbf{V} + d(\omega^k\cdot\mathbf{V})\,, \tag{18}$$

where $d_{\mathscr{T}}$ *and* d *are exterior derivatives respectively on the trajectory manifold and on the placement manifold.*

Corollary 3.1 (Spatial homotopy formula). *Under feasibility of the split* $\mathscr{L}_{\mathbf{V}}\,\omega^k = \mathscr{L}_{\mathbf{Z}}\,\omega^k + \mathscr{L}_{\mathbf{v}}\,\omega^k$*, the* LIE*-derivative* $\mathscr{L}_{\mathbf{v}}\,\omega^k$ *of a spatial form* ω^k *along the spatial motion, is expressed in terms of exterior derivatives by the spatial homotopy formula*

$$\mathscr{L}_{\mathbf{v}}\,\omega^k = (d\omega^k)\cdot\mathbf{v} + d(\omega^k\cdot\mathbf{v})\,, \tag{19}$$

where d *is the exterior derivative on a placement manifold.*

3.2 HELMHOLTZ FORMULA

As a direct consequence of the homotopy formula we derive here a noteworthy formula, due to H. VON HELMHOLTZ, that provides the expression of the time rate of variation of the flux of a spatial vector field $\mathbf{u}$ across an outer oriented material surface Σ_{OUT}, with transversal normal $\mathbf{n}$, drifted by the motion.

Preliminarily we recall the definitions of vector cross product and of usual differential operators considered in vector field theories, in terms of exterior differential.

$$\begin{aligned}
&\textit{cross product}: && \mathbf{u}\times\mathbf{v} && = \mu\cdot\mathbf{u}\cdot\mathbf{v}\,, && (\dim\mathscr{E}(t)=2)\\
&\textit{cross product}: && \mathbf{g}\cdot(\mathbf{u}\times\mathbf{v}) && = \mu\cdot\mathbf{u}\cdot\mathbf{v}\,, && (\dim\mathscr{E}(t)=3)\\
&\textit{gradient}: && df && = \mathbf{g}\cdot\nabla f\,, && (\dim\mathscr{E}(t)=2,3)\\
&\textit{curl}: && d\,(\mathbf{g}\mathbf{v}) && = (\mathrm{rot}\,\mathbf{v})\,\mu\,, && (\dim\mathscr{E}(t)=2)\\
&\textit{curl}: && d\,(\mathbf{g}\mathbf{v}) && = \mu\cdot(\mathrm{rot}\,\mathbf{v})\,, && (\dim\mathscr{E}(t)=3)\\
&\textit{divergence}: && d\,(\mu\mathbf{v}) && = (\mathrm{div}\,\mathbf{v})\mu\,. && (\dim\mathscr{E}(t)=2,3)
\end{aligned} \tag{20}$$

Setting $\omega^2 := \mu\cdot\mathbf{u} = \mathbf{g}(\mathbf{u}\,,\mathbf{n})\,\mu_\Sigma$ with $\mu_\Sigma = \mu\cdot\mathbf{n}$, we apply the split $\mathscr{L}_{\mathbf{V}}\,\omega^k = \mathscr{L}_{\mathbf{Z}}\,\omega^k + \mathscr{L}_{\mathbf{v}}\,\omega^k$ and Eq. (19) to get

$$\begin{aligned}
\partial_{\alpha=0}\int_{\varphi_\alpha(\Sigma_{\mathrm{OUT}})}\omega^2 &= \int_{\Sigma_{\mathrm{OUT}}}\mathscr{L}_{\mathbf{Z}}\,\omega^2 + \mathscr{L}_{\mathbf{v}}\,\omega^2\\
&= \int_{\Sigma_{\mathrm{OUT}}}\mathscr{L}_{\mathbf{Z}}\,\omega^2 + d(\omega^2\cdot\mathbf{v}) + (d\omega^2)\cdot\mathbf{v}\,.
\end{aligned} \tag{21}$$

To translate into the language of vector analysis we recall that

$$\begin{aligned}
\mu\cdot\mathbf{u}\cdot\mathbf{v} &= \mathbf{g}\cdot(\mathbf{u}\times\mathbf{v})\,,\\
d(\mathbf{g}\cdot(\mathbf{u}\times\mathbf{v})) &= \mu\cdot(\mathrm{rot}\,(\mathbf{u}\times\mathbf{v}))\,,\\
d(\omega^2\cdot\mathbf{v}) &= d(\mu\cdot\mathbf{u}\cdot\mathbf{v}) = d(\mathbf{g}\cdot(\mathbf{u}\times\mathbf{v})) = \mu\cdot(\mathrm{rot}\,(\mathbf{u}\times\mathbf{v}))\,,\\
(d\omega^2)\cdot\mathbf{v} &= d(\mu\cdot\mathbf{u})\cdot\mathbf{v} = (\mathrm{div}\,\mathbf{u})\,\mu\cdot\mathbf{v}\,,
\end{aligned} \tag{22}$$

and observe that from $\mathscr{L}_{\mathbf{Z}}\,\mu=\mathbf{0}$ it follows that

$$\mathscr{L}_{\mathbf{Z}}\,\omega^2 = \mathscr{L}_{\mathbf{Z}}\,(\mu\cdot\mathbf{u}) = \mu\cdot(\mathscr{L}_{\mathbf{Z}}\,\mathbf{u})\,. \tag{23}$$

Substituting in Eq. (21) and setting $\dot{\mathbf{u}} := \mathscr{L}_{\mathbf{Z}}\,\mathbf{u}$, we get HELMHOLTZ's formula

$$\partial_{\alpha=0}\int_{\varphi_\alpha(\Sigma_{\mathrm{OUT}})}\mathbf{g}(\mathbf{u},\mathbf{n})\,\mu_\Sigma = \int_{\Sigma_{\mathrm{OUT}}}\mu\cdot(\dot{\mathbf{u}}+\mathrm{rot}\,(\mathbf{u}\times\mathbf{v})+(\mathrm{div}\,\mathbf{u})\,\mathbf{v})\,. \tag{24}$$

3.3 ELECTROMAGNETIC FIELDS

Let $\mathbf{g}$ be the time invariant metric tensor field in the spatial slices and μ the associated volume form that takes a unitary value on spatial cubes with sides of unit length.

The *inner* one-form electric field $\omega^1_{\mathbf{E}}$, the *inner* one-form magnetic potential $\omega^1_{\mathbf{A}}$ and the *inner* two-form magnetic vortex $\omega^2_{\mathbf{B}} = d\omega^1_{\mathbf{A}}$ are expressed in terms of the corresponding *inner* electric vector field $\mathbf{E}$, *inner* magnetic vector field $\mathbf{A}$, and *outer* vector field of magnetic induction $\mathbf{B}$, by

$$\begin{aligned}
\omega^1_{\mathbf{E}} &= \mathbf{g}\cdot\mathbf{E}\,,\\
\omega^1_{\mathbf{A}} &= \mathbf{g}\cdot\mathbf{A}\,,\\
\omega^2_{\mathbf{B}} &= \mu\cdot\mathbf{B}\,,
\end{aligned} \tag{25}$$

The *outer* one-form magnetic winding $\omega^1_{\mathbf{H}}$, the *outer* two-form electric induction flux $\omega^2_{\mathbf{D}}$, the *outer* one-form electric current winding $\omega^1_{\mathbf{J}}$, the *outer* two-form electric current flux $\omega^2_{\mathbf{J}} = d\omega^1_{\mathbf{J}}$ are expressed in terms of the corresponding *outer* magnetic vector field $\mathbf{H}$, *inner* electric induction vector field $\mathbf{D}$, *outer* electric current vector potential $\mathbf{A_J}$, *inner* electric current vector field $\mathbf{J}$, and of the *inner* scalar electric charge ρ and the *outer* three-form electric charge density ω^3_ρ, by

$$\begin{aligned}
&\omega^1_{\mathbf{H}} = \mathbf{g}\cdot\mathbf{H}\,, &&\\
&\omega^2_{\mathbf{D}} = \mu\cdot\mathbf{D}\,, && d\omega^2_{\mathbf{D}} = \omega^3_\rho = \rho\,\mu\,,\\
&\omega^1_{\mathbf{J}} = \mathbf{g}\cdot\mathbf{A_J}\,, && \omega^2_{\mathbf{J}} = \mu\cdot\mathbf{J}\,.
\end{aligned} \tag{26}$$

The electromagnetic constitutive relations are then expressed by the following relations

$$\mathbf{B} = \mu_0(\mathbf{H}), \qquad \mathbf{D} = \varepsilon_0(\mathbf{E}), \tag{27}$$

where μ_0 and ε_0 are suitable constitutive functions. The former relates the *outer* fields $\mathbf{B}$ and $\mathbf{H}$, while the latter relates the *inner* fields $\mathbf{D}$ and $\mathbf{E}$.

From the formulae in Eq. (22) we have the following correspondences.

$$\begin{aligned}
\omega^2_{\mathbf{B}} = d\omega^1_{\mathbf{A}} \quad &\Longleftrightarrow \quad \mathbf{B} = \operatorname{rot}\mathbf{A}, \\
d\omega^2_{\mathbf{B}} = dd\omega^1_{\mathbf{A}} = \mathbf{0} \quad &\Longleftrightarrow \quad \operatorname{div}\operatorname{rot}\mathbf{A} = 0, \\
\omega^2_{\mathbf{J}} = d\omega^1_{\mathbf{J}} \quad &\Longleftrightarrow \quad \mathbf{J} = \operatorname{rot}\mathbf{A}_{\mathbf{J}}, \\
d\omega^2_{\mathbf{J}} = dd\omega^1_{\mathbf{J}} = \mathbf{0} \quad &\Longleftrightarrow \quad \operatorname{div}\operatorname{rot}\mathbf{A}_{\mathbf{J}} = 0.
\end{aligned} \tag{28}$$

4 ELECTRIC INDUCTION

4.1 STATE OF THE ART

FARADAY's law of electromagnetic induction due to a variable magnetic vortex along a space-time motion, is classically stated as a *flux rule* (Feynman et al., 1964, II.17-1), as reproduced below in Eq. (36).

The validity of this rule as a general law of Physics has been however questioned since long ago, due to difficulties in providing a clear interpretation to simple induction phenomena.

In *The Feynman Lectures on Physics*, Feynman, Leighton and Sands (1964, II.17-1), in illustrating FARADAY law of induction, say:

We know of no other place in physics where such a simple and accurate general principle requires for its real understanding an analysis in terms of two different phenomena. Usually such a beautiful generalization is found to stem from a single deep underlying principle. Nevertheless, in this case there does not appear to be any such profound implication. We have to understand the rule as the combined effect of two quite separate phenomena.

Moreover in (Feynman et al., 1964, ch. II.17-2), in commenting the difficulties emerging from applying the flux rule to FARADAY disk and to a circuit closed by rocking contacts, envisaged for discussing the applicability of FARADAY law of magnetic induction, he says that:

The "flux rule" does not work in this case. It must be applied to circuits in which the material of the circuit remains the same. When the material of the circuit is changing, we must return to the basic laws. The correct physics is always given by the two basic laws $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ *and* $\operatorname{rot}\mathbf{E} = -\dot{\mathbf{B}}$. Here ($\dot{\mathbf{B}} := \mathscr{L}_{\mathbf{Z}}\mathbf{B}$).

We may see that the way out envisaged by FEYNMAN consists in the following suggestions.

1. To abandon the flux rule as a general foundational principle for the theory of electromagnetic induction.
2. To accept that the force on the unit electric charge is given by the addition of a *transformer* electric field $\mathbf{E}$ and of a LORENTZ force field $\mathbf{v} \times \mathbf{B}$. The former electric field fulfils the induction law for a fixed unit charge, while the latter depends linearly on the spatial velocity of the unit charge.

It is clear that FEYNMAN himself was not really satisfied by the lack of elegance of these ”basic” laws.

Here an unwritten principle of Physics shows its power: *Beauty and effectiveness must go together in assessing general rules*.

4.2 OPEN QUESTIONS

Prior to illustrating the way out proposed here, let us make some introductory considerations.

FEYNMAN's point of view was shared by everyone engaged in electromagnetic theory.

Item 1) in FEYNMAN's suggestion was motivated by the difficulty of interpreting some experimental evidence in simple induction phenomena. Moreover, it is certainly not at all clear how to evaluate the *flux* of the magnetic induction and its time variation, in many technically important applications.

A simple and important instance is provided by an inductive coil in which an isolated conductive wire is winded in a complex way around a cylinder. What is there the surface through which the magnetic flux is to be evaluated?

Item 2) in FEYNMAN's suggestion is not clearly motivated and clashes against an evident deficiency. The field providing the LORENTZ force term depends on the observer appointed of measurements of time rate $\dot{\mathbf{B}}$ and of spatial velocity $\mathbf{v}$ of the moving charge.

What is more, the whole expression of the electric force $\mathbf{F}$ acting on the charge is not independent of a GALILEI change of observer, a fact that is physically not acceptable.

In the Sect. II.13-6 of (Feynman et al., 1964), entitled *The relativity of magnetic and electric fields*, FEYNMAN writes:

When we said that the magnetic force on a charge was proportional to its velocity, you may have wondered: ”What velocity? With respect to which reference frame?” It is, in fact, clear from the definition of $\mathbf{B}$ *given at the beginning of this chapter that what this vector is will depend on what we choose as a reference frame for our specification of the velocity of charges. But we have said nothing about which is the proper frame for specifying the magnetic field.*

FEYNMAN's was then well aware of this difficulty and strived to find a way out also in this case by resorting to the transformation rule of electromagnetic fields in special relativity. The same relativity argument was adduced later by Purcell (1965, ch.5).

It is certainly surprising that the understanding of simple induction phenomena must require relativity arguments also when the involved speeds are extremely lower than that of light.

The strangeness of this motivation has been confirmed by a revision of the transformation rule of electromagnetic fields in special relativity (Romano G., 2013).

Anyway, the relativity arguments adduced by FEYNMAN and PURCELL deal formally only with the LORENTZ force term and do not resolve the indeterminacy of the time rate $\dot{\mathbf{B}}$ (at a fixed spatial point) which remains a term dependent on the observer.

4.3 THE NEW RULE OF ELECTRIC INDUCTION

We show here that overcoming the exposed difficulties in the theory of electromagnetic induction is feasible without abandoning the classical framework, which is so useful and familiar to electrical engineers, and that this result can be achieved by means of a physically clear modification of the governing rules.

The new rules are elegant and able to resolve all long lasting troubles. FARADAY's rule is only a special case of a more general balance rule involving integrals along any spatial path.

No surface to evaluate the flux of the magnetic induction is needed, no closed loop to evaluate the electromotive force (E.M.F.) is needed, and an observer independent differential expression for the electric field is provided.

Applications to challenging induction phenomena classically exposed in literature, such as FARADAY's disc (BARLOW's wheel), the *homopolar* generator and HERING's experiment, see e.g. (Lehner, 2010) show that effective interpretations can be given by the new theory.

On the other hand a correction to standard formulae, by a factor one-half, must be made. Among induction phenomena that need this revision we quote the HALL effect (Hall, 1879) and the *railgun* functioning.

Guided by the previous considerations about the difficulties involved in detecting a surface for application of the *flux rule*, and by the evidence of BIOT-SAVART and AMPÈRE laws concerning the magnetic field induced by a current carrying rectilinear wire and the mutual forces exerted by current carrying wires, we introduce the new *Electric Induction Law* (E.I.L.) and *Magnetic Induction Law* (M.I.L.), which are balance laws involving arbitrary curvilinear paths undergoing motions which are required to be only *piecewise regular*.

The latter will be treated in Sect. 5.

The *Electric Induction Law* law is more general than FARADAY's flux rule and reduces to it for circuits (i.e. closed paths) undergoing *regular* motions. To simplify the statement we set the following.

Definition 4.1. *The electromotive force* $\text{E.M.F.}(L_{\text{INN}})$ *along an inner oriented material path* L_{INN} *is the sum of bulk and boundary contributions*

$$\text{E.M.F.}(L_{\text{INN}}) := \int_{L_{\text{INN}}} \omega^1_{\mathbf{E}} + \oint_{\partial L_{\text{INN}}} P\,, \tag{29}$$

where the induced electric field $\omega^1_{\mathbf{E}}$ *is an inner one-form and the electrostatic scalar potential* P *is an inner zero-form fulfilling* COULOMB*'s inverse-square force law.*

Definition 4.2. *The magnetic momentum along an inner oriented spatial path* L_{INN} *is the line-integral of the magnetic induction*

$$\text{M.M.}(L_{\text{INN}}) := \int_{\varphi_\alpha(L_{\text{INN}})} \omega^1_{\mathbf{A}}\,. \tag{30}$$

Remark 4.1. *The magnetic one-form* $\omega^1_{\mathbf{A}}$ *and the associated vector potential* $\mathbf{A}$ *are related by* $\omega^1_{\mathbf{A}} = \mathbf{g}\cdot\mathbf{A}$ *so that*

$$\varphi_\alpha\!\downarrow\!\omega^1_{\mathbf{A}} = \varphi_\alpha\!\downarrow\!(\mathbf{g}\cdot\mathbf{A}) = (\varphi_\alpha\!\downarrow\!\mathbf{g})\cdot(\varphi_\alpha\!\downarrow\!\mathbf{A})\,, \tag{31}$$

and by LEIBNIZ *rule*

$$\begin{aligned} \mathscr{L}_{\mathbf{V}}\,\omega^1_{\mathbf{A}} = \partial_{\alpha=0}\,\varphi_\alpha\!\downarrow\!\omega^1_{\mathbf{A}} &= (\mathscr{L}_{\mathbf{V}}\,\mathbf{g})\cdot\mathbf{A} + \mathbf{g}\cdot(\mathscr{L}_{\mathbf{V}}\,\mathbf{A}) \\ &= \mathbf{g}\cdot\text{EUL}(\mathbf{v})\cdot\mathbf{A} + \mathbf{g}\cdot(\mathscr{L}_{\mathbf{V}}\,\mathbf{A})\,, \end{aligned} \tag{32}$$

where $\mathscr{L}_{\mathbf{V}}\,\mathbf{g} = \mathbf{g}\cdot\text{EUL}(\mathbf{v})$*, being* $\text{EUL}(\mathbf{v}) = \text{sym}\,\nabla\mathbf{v}$ *the* EULER *stretching formula of Continuum Mechanics, expressed in terms of the* LEVI-CIVITA *connection* ∇.

Principle 4.1 (Electric induction law). *Along inner oriented material paths* L_{INN} *dragged by a piecewise regular space-time motion* $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}_{\mathscr{E}}\,;\mathscr{T}_{\mathscr{E}})$*, the time rate of the magnetic momentum is opposite to of the sum of bulk and boundary electromotive forces*

$$\text{E.M.F.}(L_{\text{INN}}) := \int_{L_{\text{INN}}} \omega^1_{\mathbf{E}} + \oint_{\partial L_{\text{INN}}} P = -\,\partial_{\alpha=0} \int_{\varphi_\alpha(L_{\text{INN}})} \omega^1_{\mathbf{A}}\,. \tag{33}$$

Applying the LIE-REYNOLDS *formula* (9) *and localizing, from* (32)*, recalling that* $dP = \mathbf{g}\cdot\nabla P$*, we get the differential law*

$$\begin{aligned} -\,\omega^1_{\mathbf{E}} &= \mathscr{L}_{\mathbf{V}}\,\omega^1_{\mathbf{A}} + dP \quad \iff \\ -\,\mathbf{E} &= \mathscr{L}_{\mathbf{V}}\,\mathbf{A} + \text{EUL}(\mathbf{v})\cdot\mathbf{A} + \nabla P\,. \end{aligned} \tag{34}$$

Proposition 4.1 (Split of differential electric induction law). *Under feasibility of the split* $\mathscr{L}_{\mathbf{V}}\,\omega^1_{\mathbf{A}} = \mathscr{L}_{\mathbf{Z}}\,\omega^1_{\mathbf{A}} + \mathscr{L}_{\mathbf{v}}\,\omega^1_{\mathbf{A}}$, *the differential law* (34) *may be formulated by the alternative expression*

$$\begin{aligned} -\,\omega^1_{\mathbf{E}} &= \mathscr{L}_{\mathbf{Z}}\,\omega^1_{\mathbf{A}} + \omega^2_{\mathbf{B}}\cdot\mathbf{v} + d(\omega^1_{\mathbf{A}}\cdot\mathbf{v}) + dP \quad \Longleftrightarrow \\ -\,\mathbf{E} &= \mathscr{L}_{\mathbf{Z}}\,\mathbf{A} + \mathbf{B}\times\mathbf{v} + \nabla(\mathbf{g}(\mathbf{A}\,,\mathbf{v})) + \nabla P\,. \end{aligned} \tag{35}$$

Proof. The result is a direct consequence of Eq. (19) in Corol. 3.1. ■

Proposition 4.2 (Faraday's flux rule). *Assuming that the path* $L_{\mathrm{INN}} = \partial\Sigma_{\mathrm{INN}}$ *is the boundary of an inner oriented surface* Σ_{INN} *undergoing a regular motion, the integral* E.I.L. *reduces to* FARADAY*'s* flux rule*:*

$$-\oint_{\partial\Sigma_{\mathrm{INN}}} \omega^1_{\mathbf{E}} = \partial_{\alpha=0}\int_{\varphi_\alpha(\Sigma_{\mathrm{INN}})} \omega^2_{\mathbf{B}}\,, \tag{36}$$

equivalent to the differential law

$$\begin{aligned} -\,d\omega^1_{\mathbf{E}} &= \mathscr{L}_{\mathbf{V}}\,\omega^2_{\mathbf{B}} \quad \Longleftrightarrow \\ -\,\mathrm{rot}\,\mathbf{E} &= \mathscr{L}_{\mathbf{V}}\,\mathbf{B} + \mathrm{TR}(\mathrm{EUL}(\mathbf{v}))\cdot\mathbf{B}\,. \end{aligned} \tag{37}$$

Proof. Being $\partial L_{\mathrm{INN}} = \partial\partial\Sigma_{\mathrm{INN}} = 0$ and setting $\omega^2_{\mathbf{B}} := d\omega^1_{\mathbf{A}}$ we get

$$\oint_{\varphi_\alpha(\partial\Sigma_{\mathrm{INN}})} \omega^1_{\mathbf{A}} = \oint_{\partial\varphi_\alpha(\Sigma_{\mathrm{INN}})} \omega^1_{\mathbf{A}} = \int_{\varphi_\alpha(\Sigma_{\mathrm{INN}})} d\omega^1_{\mathbf{A}} = \int_{\varphi_\alpha(\Sigma_{\mathrm{INN}})} \omega^2_{\mathbf{B}}\,, \tag{38}$$

and the E.I.L. (33) gives Eq. (36). Independence of the choice of the surface Σ_{INN} such that $L_{\mathrm{INN}} = \partial\Sigma_{\mathrm{INN}}$ holds provided that the chain of two such surfaces is the boundary of a 3D manifold $\mathrm{V}_{\mathrm{INN}}$. Then, by the assumption that $d\omega^2_{\mathbf{B}} = \mathbf{0}$ (there are no magnetic monopoles), we get

$$\oint_{\varphi_\alpha(\partial\mathrm{V}_{\mathrm{INN}})} \omega^2_{\mathbf{B}} = \int_{\varphi_\alpha(\mathrm{V}_{\mathrm{INN}})} d\omega^2_{\mathbf{B}} = 0\,. \tag{39}$$

The differential Eq. $(37)_1$ is deduced by localisation and Eq. $(37)_2$ comes from

$$\mathscr{L}_{\mathbf{V}}\,\omega^2_{\mathbf{B}} = \mathscr{L}_{\mathbf{V}}\,(\mu\cdot\mathbf{B}) = (\mathscr{L}_{\mathbf{V}}\,\mu)\cdot\mathbf{B} + \mu\cdot\mathscr{L}_{\mathbf{V}}\,(\mathbf{B})\,, \tag{40}$$

with $\mathscr{L}_{\mathbf{V}}\,\mu = \mathrm{TR}(\mathrm{EUL}(\mathbf{v}))\,\mu$ since the volumetric stretching is the linear invariant of the EULER stretching tensor. ■

Remark 4.2. *The standard formulation of Eq.* (36) *cannot be directly applied to induction phenomena in which the material path L_{INN} is not the boundary of a surface Σ_{INN} (this is a most usual situation in applications, as for instance in evaluating the* E.M.F. *generated in a coil immersed in a magnetic induction field). Moreover the usual denomination of* flux rule *should be changed into* vorticity rule *to conform to the physically consistent assumption of an inner oriented surface, and to* MAXWELL*'s point of view.*

Effectiveness of the general expression of the E.I.L. provided by Eq. (33) and Eq. (34) will be checked in Sect. 6, versus experimental evidence of FARADAY unipolar motor/generator and of conductive bar sliding on rails in a field of magnetic induction, whose interpretation on the basis of the flux rule Eq. (36) is troublesome, as reported in recent literature (Lehner, 2010).

4.4 HISTORICAL NOTES

The expression (35) of the differential E.I.L. is coincident with the one contributed in cartesian coordinates by J.J. Thomson (1893, Ch.VII, "Electromotive intensity in moving bodies", p.534). This was the first (and to the author's knowledge, also the last) appearance of the terms $d\langle \omega^1_{\mathbf{A}}, \mathbf{v} \rangle$. This term was absent in the previous treatments by Clerk-Maxwell (1861, 1865, 1873) because, according to J.J. THOMSON, the scalar field $\langle \omega^1_{\mathbf{A}}, \mathbf{v} \rangle$ was there merged with the electrostatic potential P into a global scalar potential Ψ.

The induction law formulated in (Clerk-Maxwell, 1861, (77) p. 342) and (Clerk-Maxwell, 1865, (D) p. 485) writes

$$\begin{aligned} -\,\omega^1_{\mathbf{E}} = \mathscr{L}_{\mathbf{Z}}\, \omega^1_{\mathbf{A}} + \omega^2_{\mathbf{B}} \cdot \mathbf{v} + d\Psi \quad &\Longleftrightarrow \\ -\,\mathbf{E} = \dot{\mathbf{A}} + \mathbf{B} \times \mathbf{v} + \nabla\Psi\,, \quad &(\dot{\mathbf{A}} := \mathscr{L}_{\mathbf{Z}}\mathbf{A})\,. \end{aligned} \tag{41}$$

Eq. (41) shows that the force term $\mathbf{v} \times \mathbf{B}$, usually named after LORENTZ, was introduced about thirty years before by MAXWELL who formulated the expression of the magnetically induced electric field in terms of the vector potential $\mathbf{A}$. The term is in fact also reported by Hertz (1892) as a well-known result.

In the subsequent literature the scalar potential Ψ was wrongly identified with the electrostatic potential and hence, due to the lack of the velocity depending term $d\langle \omega^1_{\mathbf{A}}, \mathbf{v} \rangle$, the expression of the electric field provided by Eq. (41) became observer dependent.

What is more, according to simplifications introduced by Heaviside (1892) and by Hertz (1892) about the end of the nineteenth century, the induction law was expressed in terms of the sole magnetic induction field $\mathbf{B} = \operatorname{rot}\mathbf{A}$.

Accordingly, Eq. (41) was written, setting $\dot{\mathbf{B}} := \mathscr{L}_{\mathbf{Z}} \mathbf{B}$, as an equality in terms of rotors

$$- \operatorname{rot} \mathbf{E} = \dot{\mathbf{B}} + \operatorname{rot} (\mathbf{B} \times \mathbf{v}) . \tag{42}$$

Since scalar potentials vanish in Eq. (42), the differences between Eq. (35) and (41) resulted to be completely obfuscated.

This was a major drawback since the expression Eq. (35), in terms of the vector potential $\mathbf{A}$, is the one fulfilling the basic property of observer independence of the electric field. This is readily checked by Eq. $(34)_1$ on the basis of the naturality properties Eq. (16) and (13).

As a consequence, the inductive E.M.F. was since then universally considered to stem from two distinct sources, see e.g. Refs. (Weyl, 1922), (Panofski and Phillips, 1962), (Post, 1962), (Feynman et al., 1964), (Barut, 1980), (Purcell, 1965), (Landau and Lifshits, 1987), (Jackson, 1999), (Kovetz, 2000), (Wegner, 2003), (Lehner, 2010), (Sadiku, 2010).

1. The first source, due to variability of magnetic induction with time in a circuit at rest, is the time-derivative at fixed spatial position (the *transformer* term).
$$- \mathscr{L}_{\mathbf{Z}} \, \omega^1_{\mathbf{A}} = - \mathbf{g} \cdot \mathscr{L}_{\mathbf{Z}} \mathbf{B} \tag{43}$$
2. The second source, due to motion of the charge in presence of the magnetic vortex field, is evaluated by the one-form $-\omega^2_{\mathbf{B}} \cdot \mathbf{v} = \mathbf{g}(\mathbf{v} \times \mathbf{B})$ (the *motional* term).

This latter term is also usually referred to as "LORENTZ *force*", a name still widespread in literature on electromagnetic induction, but with a faulty motivation.

The motivation why the *motional* term was *not* attributed to MAXWELL but to LORENTZ, founds its roots in the transformation rules for electromagnetic fields due to changes of observer assumed by Lorentz (1904), Einsten (1905) and Minkowski (1908).

Space-time frame transformations, corresponding to relative translational motions which preserve the speed c of light in *vacuo* were named after LORENTZ by Poincaré (1906), but, according to Minkowski (1908), the same transformation was conceived about two decades before by Voigt (1887).

The analysis this frame transformation performed originally in (Lorentz, 1904) and reported in (Einsten, 1905), led to the conclusion that the electric field would be modified in its component transversal to the relative motion, by the addition of a term $\mathbf{w} \times \mathbf{B}$, where $\mathbf{w}$ is the relative velocity evaluated by the observer who is performing the measurements, and by an increase according to the relativistic factor

$$\gamma := (1 - w^2/c^2)^{-1/2}\,, \tag{44}$$

where $w = \sqrt{\mathbf{g}(\mathbf{w}\,,\mathbf{w})}$ is the relative speed.

The transformation of parallel and transversal components of the electric field, assumed by the treatments of special relativity exposed in (Lorentz, 1904), (Einsten, 1905), (Minkowski, 1908), is given by

$$\begin{cases} \mathbf{E}^{\|} \mapsto \mathbf{E}^{\|}\,, \\ \mathbf{E}^{\perp} \mapsto \gamma(\mathbf{E}^{\perp} + \mathbf{w} \times \mathbf{B})\,. \end{cases} \tag{45}$$

Clearly, the additive term $\mathbf{w} \times \mathbf{B}$ in Eq. $(45)_2$ does not vanish in the classical limit $\gamma \to 1$ and would therefore also be generated by a GALILEI change of frame. This unreliable conclusion should convince that some misstatements occurred.

Moreover, under the action of a LORENTZ frame transformation, a spatial electric field is pushed into a space-time field that is no more a spatial field, in the same reference frame. Indeed LORENTZ transformations do not preserve simultaneity of events.

Accordingly, in a relativistic context, a full space-time formulation of electromagnetics should be adopted, as first conceived in (Hargreaves, 1908; Bateman, 1910; E. Cartan, 1924), reported in (Truesdell and Toupin, 1960; Misner et al., 1973) and revised in (Romano G., 2013).

The space-time formulation of electromagnetic induction, requires concepts and methods of Differential Geometry, with special regard to exterior calculus of differential forms.

As a matter of fact, the differential geometric analysis recently performed in (Romano G., 2013), reveals that the transformation expressed by Eq. (45) is the outcome of a wrong evaluation of the way in which the electric field transforms under a change of frame.

According to the results exposed in (Romano G., 2013) the transversal component of the electric field is unchanged by the LORENTZ transformation, while the parallel component is changed, as follows

$$\begin{cases} \mathbf{E}^{\|} \mapsto \gamma(\mathbf{E}^{\|} - \mathbf{g}(\mathbf{v}/c\,,\mathbf{E})\,\mathbf{w}/c)\,, \\ \mathbf{E}^{\perp} \mapsto \mathbf{E}^{\perp}\,, \end{cases} \tag{46}$$

where $\mathbf{v}$ is the spatial velocity of the body, a term which was assumed to vanish in previous treatments leading to Eq. (45). The electric field results to be *not* changed by LORENTZ transformations in the classical limit $\gamma \to 1$, and hence also by GALILEI transformations, as expected on physical ground.

4.5 FRAME INVARIANCE

The troubles concerning what observer is measuring the spatial velocity $\mathbf{v}$, clearly exposed in (Feynman et al., 1964, II.13-6), are overcome by adding to Eq. (41) the missing term $d\langle \omega^1_{\mathbf{A}}, \mathbf{v} \rangle$ included in the complete expression Eq. (35).

Observer-invariance of the E.I.L. is then proven on the basis of the naturality properties Eqs. (13) and (16) of the differential dP and of the LIE derivative $\mathscr{L}_{\mathbf{V}}\, \omega^1_{\mathbf{A}}$.

Indeed, observer-invariance of the electromagnetic fields, under (simultaneity preserving) EUCLID change of frame $\zeta : \mathscr{T} \mapsto \mathscr{T}_\zeta$, is assumed as a basic axiom of the theory.

This means that for the one-forms $\omega^1_{\mathbf{E}}$ and $\omega^1_{\mathbf{A}}$ and for the scalar potential P we have

$$(\omega^1_{\mathbf{E}})_\zeta = \zeta{\uparrow}\omega^1_{\mathbf{E}} \,, \quad (\omega^1_{\mathbf{A}})_\zeta = \zeta{\uparrow}\omega^1_{\mathbf{A}} \,, \quad (P)_\zeta = \zeta{\uparrow}P \,, \tag{47}$$

so that the differential E.I.L. fulfils the frame invariance property

$$\begin{aligned} -\,\omega^1_{\mathbf{E}} &= \mathscr{L}_{\mathbf{V}}\, \omega^1_{\mathbf{A}} + dP \quad \Longleftrightarrow \\ -\,\zeta{\uparrow}\omega^1_{\mathbf{E}} &= \zeta{\uparrow}(\mathscr{L}_{\mathbf{V}}\, \omega^1_{\mathbf{A}}) + \zeta{\uparrow}(dP) \quad \Longleftrightarrow \\ -\,\zeta{\uparrow}\omega^1_{\mathbf{E}} &= \mathscr{L}_{(\zeta\uparrow\mathbf{V})}\, (\zeta{\uparrow}\omega^1_{\mathbf{A}}) + d_\zeta(\zeta{\uparrow}P) \quad \Longleftrightarrow \\ -\,(\omega^1_{\mathbf{E}})_\zeta &= \mathscr{L}_{\mathbf{V}_\zeta}\, ((\omega^1_{\mathbf{A}})_\zeta) + d_\zeta P_\zeta \,. \end{aligned} \tag{48}$$

5 MAGNETIC INDUCTION

The MAXWELL-AMPÈRE's law of electromagnetic induction of a magneto-motive force (M.M.F.), due to an electric current flux, is classically stated as a *flux rule*, see Eq. (54) below.

Let us consider, a region where free electric charges are absent (i.e. $\omega^3_\rho = \mathbf{0}$) and there are no sources of electric current (i.e. $d\omega^2_{\mathbf{J}} = \mathbf{0}$).

We show that therein the MAXWELL-AMPÈRE's law can be formulated as a balance law for a magnetic winding *outer* one-form $\omega^1_{\mathbf{H}}$, involving the electric induction *outer* one-form $\omega^1_{\mathbf{D}}$ and the electric current potential *outer* one-form $\omega^1_{\mathbf{J}}$, which we will call the Magnetic Induction Law (M.I.L.).

By GAUSS law

$$\omega^3_\rho = d\omega^2_{\mathbf{D}} \,, \tag{49}$$

and by assumption $\omega^3_\rho = \mathbf{0}$ and $d\omega^2_{\mathbf{J}} = \mathbf{0}$.

Then, by POINCARÉ lemma exposed in Sect. 2, we get

$$\omega^2_{\mathbf{D}} = d\omega^1_{\mathbf{D}} \,, \qquad \omega^2_{\mathbf{J}} = d\omega^1_{\mathbf{J}} \,. \tag{50}$$

Principle 5.1 (Magnetic induction Law). *Around an outer oriented spatial path dragged by a piecewise regular space-time motion* $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}_\mathscr{E}; \mathscr{T}_\mathscr{E})$, *the induced magnetomotive force (*M.M.F.*) is equal to the sum of the time rate of the integral of the electric winding* $\omega^1_{\mathbf{D}}$ *and of the integral of the electric current potential* $\omega^1_{\mathbf{J}}$

$$\text{M.M.F.}(L_{\text{OUT}}) := \int_{L_{\text{OUT}}} \omega^1_{\mathbf{H}} = \partial_{\alpha=0} \int_{\varphi_\alpha(L_{\text{OUT}})} \omega^1_{\mathbf{D}} + \int_{L_{\text{OUT}}} \omega^1_{\mathbf{J}}\,. \tag{51}$$

Applying the LIE-REYNOLDS *formula Eq.* (9) *and localizing, we get the differential law*

$$\begin{aligned} \omega^1_{\mathbf{H}} &= \mathscr{L}_{\mathbf{V}}\, \omega^1_{\mathbf{D}} + \omega^1_{\mathbf{J}} \quad \Longleftrightarrow \\ \mathbf{H} &= \mathscr{L}_{\mathbf{V}}\, \mathbf{D} + \text{EUL}(\mathbf{v}) \cdot \mathbf{D} + \mathbf{A}_{\mathbf{J}}\,. \end{aligned} \tag{52}$$

Proposition 5.1 (Split of differential magnetic induction law). *Under feasibility of the split* $\mathscr{L}_{\mathbf{V}}\, \omega^1_{\mathbf{D}} = \mathscr{L}_{\mathbf{Z}}\, \omega^1_{\mathbf{D}} + \mathscr{L}_{\mathbf{v}}\, \omega^1_{\mathbf{D}}$, *the integral law* (51) *is equivalent to the frame invariant differential law*

$$\begin{aligned} \omega^1_{\mathbf{H}} &= \mathscr{L}_{\mathbf{Z}}\, \omega^1_{\mathbf{D}} + \omega^2_{\mathbf{D}} \cdot \mathbf{v} + d(\omega^1_{\mathbf{D}} \cdot \mathbf{v}) + \omega^1_{\mathbf{J}} \quad \Longleftrightarrow \\ \mathbf{H} &= \mathscr{L}_{\mathbf{Z}}\, \mathbf{A}_{\mathbf{D}} + \mathbf{D} \times \mathbf{v} + \nabla(\mathbf{g}(\mathbf{D}, \mathbf{v})) + \mathbf{A}_{\mathbf{J}}\,. \end{aligned} \tag{53}$$

Proof. The result is a direct consequence of Eq. (19) in Corol. 3.1. ■

Proposition 5.2 (Ampere's flux rule). *Assuming that the path* $L_{\text{OUT}} = \partial\Sigma_{\text{OUT}}$ *is the boundary of an outer oriented surface* Σ_{OUT} *undergoing a regular motion, the integral* M.I.L. *reduces to the standard* MAXWELL-AMPÈRE*'s* flux rule*:*

$$\oint_{\partial\Sigma_{\text{OUT}}} \omega^1_{\mathbf{H}} = \partial_{\alpha=0} \int_{\varphi_\alpha(\Sigma_{\text{OUT}})} \omega^2_{\mathbf{D}} + \int_{\Sigma_{\text{OUT}}} \omega^2_{\mathbf{J}}\,. \tag{54}$$

equivalent to the differential law

$$\begin{aligned} d\omega^1_{\mathbf{H}} &= \mathscr{L}_{\mathbf{V}}\, \omega^2_{\mathbf{D}} + \omega^2_{\mathbf{J}} \quad \Longleftrightarrow \\ \operatorname{rot} \mathbf{H} &= \mathscr{L}_{\mathbf{V}}\, \mathbf{D} + \text{EUL}(\mathbf{v}) \cdot \mathbf{D} + \mathbf{A}_{\mathbf{J}}\,. \end{aligned} \tag{55}$$

Proof. Being $\partial L_{\text{OUT}} = \partial\partial\Sigma_{\text{OUT}} = 0$ and setting $\omega^2_{\mathbf{D}} := d\omega^1_{\mathbf{D}}$ we get

$$\oint_{\varphi_\alpha(\partial\Sigma_{\text{OUT}})} \omega^1_{\mathbf{D}} = \oint_{\partial\varphi_\alpha(\Sigma_{\text{OUT}})} \omega^1_{\mathbf{D}} = \int_{\varphi_\alpha(\Sigma_{\text{OUT}})} d\omega^1_{\mathbf{D}} = \int_{\varphi_\alpha(\Sigma_{\text{OUT}})} \omega^2_{\mathbf{D}}\,, \tag{56}$$

and the M.I.L. (51) gives Eq. (54).

Independence of the choice of the surface Σ_{OUT} such that $L_{\text{OUT}} = \partial\Sigma_{\text{OUT}}$ holds provided that the chain of two such surfaces is the boundary of a 3D manifold V_{OUT}. Then, indeed, the principle of *electric charge conservation* states that

$$\partial_{\alpha=0} \int_{\varphi_\alpha(\text{V}_{\text{OUT}})} \omega^3_\rho + \oint_{\partial \text{V}_{\text{OUT}}} \omega^2_{\mathbf{J}} = 0\,. \tag{57}$$

By GAUSS law

$$\omega^3_\rho = d\omega^2_{\mathbf{D}}\,, \tag{58}$$

the conservation law in Eq. (57) may be written as

$$\partial_{\alpha=0} \oint_{\varphi_\alpha(\partial\text{V}_{\text{OUT}})} \omega^2_{\mathbf{D}} + \oint_{\partial \text{V}_{\text{OUT}}} \omega^2_{\mathbf{J}} = 0\,, \tag{59}$$

thus giving the result. ∎

Applying LIE-REYNOLDS formula Eq. (9) and STOKES formula Eq. (4) to the balance laws Eq. (57) and Eq. (59), localising and recalling the commutativity property in Eq. (11) and GAUSS law Eq. (58), we get

$$d(\mathscr{L}_{\mathbf{V}}\,\omega^2_{\mathbf{D}} + \omega^2_{\mathbf{J}}) = \mathscr{L}_{\mathbf{V}}\,d\omega^2_{\mathbf{D}} + d\omega^2_{\mathbf{J}} = \mathscr{L}_{\mathbf{V}}\,\omega^3_\rho + d\omega^2_{\mathbf{J}} = \mathbf{0}\,. \tag{60}$$

When it is allowed to split the LIE derivative into space and time directions, we get

$$\mathscr{L}_{\mathbf{Z}}\,\omega^3_\rho + \mathscr{L}_{\mathbf{v}}\,\omega^3_\rho + d\omega^2_{\mathbf{J}} = \mathbf{0}\,, \tag{61}$$

which is equivalent to due to the formula by Helmholtz (1870)

$$\mathscr{L}_{\mathbf{Z}}\,\rho + \text{div}\,(\rho\,\mathbf{v}) + \text{div}\,\mathbf{J} = 0\,. \tag{62}$$

Eq. (55) is deduced in the same way as Eq. (34).

Remark 5.1. *The standard formulation Eq.* (54) *cannot be directly applied to explain induction phenomena in which the material path* L_{OUT} *is not the boundary of a surface* Σ_{OUT}*. This is an usual situation in applications, for instance in evaluating the* M.M.F. *induced by an electric current in a coil.*

6 EXAMPLES OF APPLICATION

Our main interest is to show how the new electric induction law, exposed in Eqs. (33) and (34), is capable of providing a direct interpretation of experimental evidence not addressable, neither by the flux rule Eq. (36) nor by the LORENTZ force Eq. (41).

6.1 CHARGED BODY TRANSLATING IN A UNIFORM MAGNETIC VORTEX

Let a material body be in translational motion $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T};\mathscr{T})$ across a region in which a magnetic vortex two-form $\omega^2_{\mathbf{B}}$, is spatially constant according to the standard EUCLID connection ∇, so that

$$\nabla \omega^2_{\mathbf{B}} = 0 .$$

Let us first explain in discursive terms the idea leading to the result.

The *inner* vector potential $\mathbf{A}$ associated with the *outer* vector field $\mathbf{B}$ of magnetic induction may be assumed to have cylindrical symmetry around a longitudinal axis with linear radial distribution. The body velocity field is assumed to be orthogonal to the magnetic induction.

Accordingly, the parallel derivative of the vector potential $\mathbf{A}$ along the motion velocity, is a vector field with the direction of the vector potential and intensity given by the product of half the intensity of the rotor of $\mathbf{A}$ times the intensity of the velocity.

Taking into account the usual orientations, and evaluating the parallel derivative of the magnetic vector field $\mathbf{B}$, the electric field due to magnetic induction is given by *one-half* the standard expression of the LORENTZ force (per unit electric charge)

$$\mathbf{E} = \tfrac{1}{2}\mathbf{v} \times \mathbf{B} . \tag{63}$$

To see this result expressed in formulae, we begin by providing the expression of the LIE derivative along the spatial vector field $\mathbf{v}$ of a spatial covariant tensor field α_{COV}, in terms of parallel derivatives ∇ (Romano G., 2007)

$$\mathscr{L}_{\mathbf{v}}\, \alpha_{\mathrm{COV}} = \nabla_{\mathbf{v}}\, \alpha_{\mathrm{COV}} + \alpha_{\mathrm{COV}} \cdot \nabla \mathbf{v} + (\nabla \mathbf{v})^* \cdot \alpha_{\mathrm{COV}} , \tag{64}$$

where the star * denotes duality. This formula will be referred to in the following Lemma.

Lemma 6.1 (Linear Faraday potential). *A magnetic vortex field which is spatially constant, according to the standard connection of* EUCLID *space, so that* $\nabla \omega^2_{\mathbf{B}} = \mathbf{0}$*, admits a magnetic potential one-form* $\omega^1_{\mathbf{A}}$*, such that* $\omega^2_{\mathbf{B}} = d\omega^1_{\mathbf{A}}$*, having the linear distribution*

$$\omega^1_{\mathbf{A}} := \tfrac{1}{2}\omega^2_{\mathbf{B}} \cdot \mathbf{r} = \tfrac{1}{2}\mu \cdot \mathbf{B} \cdot \mathbf{r} . \tag{65}$$

Here μ *is the volume form associated with the metric tensor field* $\mathbf{g}$*. The vector field* $\mathbf{r} \in \mathrm{C}^1(\mathscr{S};T\mathscr{S})$ *is defined by* $\mathbf{r}(\mathbf{p}) := \mathbf{x} \in T_{\mathbf{x}}\mathscr{S}$ *for all* $\mathbf{x} = \mathbf{p} - \mathbf{o}$.

Proof. Clearly for any $\mathbf{h} \in T_{\mathbf{x}}\mathscr{S}$ we have

$$\nabla_{\mathbf{h}}\,\mathbf{r} = \lim_{\varepsilon \to 0} \varepsilon^{-1}(\mathbf{r}(\mathbf{p}+\varepsilon\mathbf{h}) - \mathbf{r}(\mathbf{p})) = \lim_{\varepsilon \to 0} \varepsilon^{-1}(\mathbf{x}+\varepsilon\mathbf{h}-\mathbf{x}) = \mathbf{h}\,, \tag{66}$$

so that $\nabla\mathbf{r} = \mathbf{I}$, $(\nabla\mathbf{r})^* = \mathbf{I}^*$ with $\mathbf{I}$ identity map on $T\mathscr{S}$.

Being $d\omega^2_{\mathbf{B}} = 0$ and by assumption $\nabla\omega^2_{\mathbf{B}} = 0$, the homotopy formula and the expression Eq. (64) of the LIE derivative in terms of parallel derivative, give

$$d(\omega^2_{\mathbf{B}} \cdot \mathbf{r}) = \mathscr{L}_{\mathbf{r}}\,\omega^2_{\mathbf{B}} = \nabla_{\mathbf{r}}\,\omega^2_{\mathbf{B}} + \omega^2_{\mathbf{B}} \cdot \nabla\mathbf{r} + (\nabla\mathbf{r})^* \cdot \omega^2_{\mathbf{B}} = 2\,\omega^2_{\mathbf{B}}\,,$$

which is the result to be proved. ■

Proposition 6.1 (Electric field in a translating body). *A body with a translational motion, across a region of spatially uniform magnetic vortex* $\omega^2_{\mathbf{B}}$*, according to the standard connection* ∇ *of* EUCLID *space, experiences an electric field given by the formula*

$$\begin{aligned}
\omega^1_{\mathbf{E}} &= -\mathscr{L}_{\mathbf{Z}}\,\omega^1_{\mathbf{A}} - \tfrac{1}{2}\,\omega^2_{\mathbf{B}} \cdot \mathbf{v} - dP \\
&= -\mathscr{L}_{\mathbf{Z}}\,\omega^1_{\mathbf{A}} + d(\omega^1_{\mathbf{A}} \cdot \mathbf{v}) - dP \quad \Longleftrightarrow \\
\mathbf{E} &= -\dot{\mathbf{A}} + \tfrac{1}{2}\,(\mathbf{v} \times \mathbf{B}) - \nabla P \\
&= -\dot{\mathbf{A}} + d\,\mathbf{g}(\mathbf{A}\,,\mathbf{v}) - \nabla P\,.
\end{aligned} \tag{67}$$

Proof. Let us consider an observer detecting a translational motion $\varphi_\alpha \in \mathrm{C}^1(\mathscr{T}_{\mathscr{E}}\,;\mathscr{T}_{\mathscr{E}})$ and measuring the space-time velocity $\mathbf{V} := \partial_{\alpha=0}\,\varphi_\alpha = \mathbf{Z}+\mathbf{v}$, whose spatial component is uniform, i.e. $\nabla\mathbf{v} = \mathbf{0}$. From the formula for the LIE derivative in terms of parallel derivatives, we get

$$\mathscr{L}_{\mathbf{v}}\,\omega^1_{\mathbf{A}} = \nabla_{\mathbf{v}}\,\omega^1_{\mathbf{A}} + \omega^1_{\mathbf{A}} \cdot \nabla\mathbf{v} = \nabla_{\mathbf{v}}\,\omega^1_{\mathbf{A}}\,.$$

Being $\nabla\omega^2_{\mathbf{B}} = 0$, Lemma 6.1 gives $\omega^1_{\mathbf{A}} = \frac{1}{2}\omega^2_{\mathbf{B}} \cdot \mathbf{r}$ and hence

$$\mathscr{L}_{\mathbf{v}}\,\omega^1_{\mathbf{A}} = \nabla_{\mathbf{v}}\,\omega^1_{\mathbf{A}} = \tfrac{1}{2}\omega^2_{\mathbf{B}} \cdot \mathbf{v}\,.$$

Then, being $\nabla P = 0$, from (34) we infer that

$$-\omega^1_{\mathbf{E}} = \mathscr{L}_{\mathbf{V}}\,\omega^1_{\mathbf{A}} = \mathscr{L}_{\mathbf{Z}}\,\omega^1_{\mathbf{A}} + \mathscr{L}_{\mathbf{v}}\,\omega^1_{\mathbf{A}} = \mathscr{L}_{\mathbf{Z}}\,\omega^1_{\mathbf{A}} + \tfrac{1}{2}\omega^2_{\mathbf{B}} \cdot \mathbf{v}\,.$$

Finally the computation

$$d(\omega^1_{\mathbf{A}} \cdot \mathbf{v}) = \tfrac{1}{2}d(\omega^2_{\mathbf{B}} \cdot \mathbf{r} \cdot \mathbf{v}) = -\tfrac{1}{2}d(\omega^2_{\mathbf{B}} \cdot \mathbf{v} \cdot \mathbf{r}) = -\tfrac{1}{2}\omega^2_{\mathbf{B}} \cdot \mathbf{v}\,, \tag{68}$$

shows that electric field admits a velocity-dependent scalar potential. ■

The one-form

$$-\tfrac{1}{2}\omega^2_{\mathbf{B}} \cdot \mathbf{v} = -\tfrac{1}{2}\mu \cdot \mathbf{B} \cdot \mathbf{v} = \tfrac{1}{2}\mathbf{g} \cdot (\mathbf{v} \times \mathbf{B}) \tag{69}$$

provides the velocity-dependent part of the electric field (*force* per unit electric charge) as detected by an observer who measures a time-invariant magnetic potential $\omega^1_{\mathbf{A}}$ at a fixed spatial position ($\mathscr{L}_{\mathbf{Z}}\,\omega^1_{\mathbf{A}} = \mathbf{0}$) and spatially uniform magnetic vortex $\omega^2_{\mathbf{B}}$ ($\nabla\omega^2_{\mathbf{B}} = 0$) and scalar electric potential P ($\nabla P = \mathbf{0}$).

Remark 6.1. *It is manifest that the so-called* LORENTZ *force law is contradicted by the previous calculation leading to the formula* $\mathbf{E} = \frac{1}{2}(\mathbf{v} \times \mathbf{B})$, *which instead agrees with the* 1881 *findings in (J.J. Thomson, 1881), later reproposed in (J.J. Thomson, 1893, Ch.VII, "Electromotive intensity in moving bodies", p.534). His result was subsequently modified by* OLIVER HEAVISIDE *in* 1885 – 1889 *and by* HEINRICH HERTZ *and* HENDRIK LORENTZ *in* 1892, *who eliminated the factor one-half, probably giving credit to the original formula Eq.* (41) *by* MAXWELL. *In (Hertz, 1892, XVI-2, p.248) the expression in Eq.* (41) *is in fact considered as well-known.*

Therein H.R. HERTZ *also provided a brief discussion and a warning against the interpretation, of the single terms there involved, as electric forces. These historical notes, taken partly from the book (Darrigol, 2000) and complemented by direct reading of the original sources, came to the attention of the author just after the present theory was independently developed. The same occurred for the original contribution by J.J. Thomson (1881) whose formula matches perfectly the new Eq.* (35), *when the space-time split is feasible.*

6.2 HERING'S EXPERIMENT

HERING's experiment, discussed in (Lehner, 2010, 6.1.4. p.349), can be interpreted according to the new rule Eq. (33) by observing that, in opening a closed circuit immersed transversally in a uniform magnetic field, there is no material line moving in the magnetic field and hence no E.M.F. is induced between the sliding contacts, as confirmed by the experimental results.

6.3 FARADAY'S DISC (BARLOW'S WHEEL)

The FARADAY disk is a classical device constructed from a brass or copper disk that can rotate in a transversal magnetic induction field. The induction E.M.F. between the axel of the disk and a point on its rim is measured by closing a circuit with the aid of brush contacts.

This failure of the *flux rule* has been recently reported also in (Lehner, 2010, 6.1.4. p.349).

As a consequence of the new rule Eq. (33), and according to the special result in Eq. (63), the radially moving charges in the material are subject to a force $\frac{1}{2}\mathbf{v}\times\mathbf{B}$, with the velocity $\mathbf{v}$ of the charges measured by an observer who detects a time-invariant field of magnetic potential. This force distribution generates a torque that pushes the rotation of the wheel.

The new theory provides thus an interpretation free of ambiguity with a torque which is one half the one expressed by the LORENTZ force term. This is in accord with the new statement in Prop. 4.2 which requires regularity of the motion.

Contrary to common claims, the *flux rule* Eq. (36) has no validity for circuits with sliding contacts. This also clarifies the difficulties of standard treatment in interpreting the result of HERING's experiment 6.2.

6.4 FARADAY'S HOMOPOLAR GENERATOR

According to Feynman et al. (1964, II.17.2): *as the disc rotates, the "circuit", in the sense of the place in space where the currents are, is always the same. But the part of the "circuit" in the disc is in material which is moving. Although the flux through the "circuit" is constant, there is still an* E.M.F.*, as can be observed by the deflection of the galvanometer. Clearly, here is a case where the* $\mathbf{v}\times\mathbf{B}$ *force in the moving disc gives rise to an* E.M.F. *which cannot be equated to a change of flux.*

Let us now discuss FARADAY *homopolar generator* by applying the new integral expression of the E.I.L. provided in Eq. (33).

If the magnetic vortex in the disk is spatially uniform and constant in time, the magnetic potential will be distributed with polar symmetry and consequently the magnetically induced electric field in the disk vanishes identically, because the time derivative at the r.h.s of (33) does vanish for any radial or circumferential path of integration.

The same conclusion may be reached by a more involved differential analysis.

The electromotive force in the circuit will be non vanishing only if the magnetic vortex field in the disk is nonuniform. This result is in sharp contrast with the conclusions in (Feynman et al., 1964, II.17.2) on the basis of LORENTZ force law, and suggests a simple experiment to confirm the new theory.

An analysis of FARADAY homopolar generator based on the flow rule is reported in (Lehner, 2010, 6.1.4. p. 350), with a doubtful conclusion about whether a fixed or a spinning radius should be considered.

Let us underline the basically different prediction of the new theory concerning FARADAY disk and FARADAY *homopolar generator*. If the conductive disc is immersed in a uniform magnetic induction field, no electromotive force is generated by spinning the disc, but, by applying an E.M.F. between sliding contacts at the axel and at the rim, a rotation of the disc will be induced.

6.5 SLIDING BAR ON RAILS UNDER A UNIFORM MAGNETIC VORTEX

Let us consider the problem concerning the electromotive force (E.M.F.) generated in a conductive bar of length $\mathbf{l}$ sliding on two fixed parallel rails under the action of a magnetic vortex which is spatially uniform, time-independent and coplanar. An observer sitting on the rails measures a time independent FARADAY potential field and may thus evaluate the E.M.F. due to the electric field distributed along the bar by integration

$$\omega^1_{\mathbf{E}} \cdot \mathbf{l} = -\tfrac{1}{2}\omega^2_{\mathbf{B}} \cdot \mathbf{v} \cdot \mathbf{l}. \tag{70}$$

On the other hand, by the integral flow rule formula, the E.M.F. should be evaluated along a circuit and hence would depend on how the circuit is closed.

This instructive problem is discussed in (Sadiku, 2010, C. *Moving Loop in Time-Varying Field*, Example 9.1, p. 375), by tacitly assuming a GALILEI observer sitting on the rails and adopting the LORENTZ force expression for the induced electric field.

The same problem with one bar fixed and another one translating on the rails is discussed in (Feynman et al., 1964, II.17.1, fig.17.1) both in terms of the *flux rule* and in terms of the LORENTZ force (also with a tacit choice of the suitable GALILEI observer).

Both analyses lead to the same value of the total E.M.F. in the circuit given by

$$\omega^1_{\mathbf{E}} \cdot \mathbf{l} = -\omega^2_{\mathbf{B}} \cdot \mathbf{v} \cdot \mathbf{l}, \tag{71}$$

thus evaluating an electromotive force which is just doubled with respect to the one predicted by the new theory. As discussed above, according to the new theory the presence of sliding contacts, and hence of discontinuous velocities, prevents any application of the *flux rule*.

6.6 THE HALL EFFECT

The HALL effect consists in detecting a potential difference (HALL voltage) on opposite sides of a thin sheet of conducting or semiconducting material through which an electric current is flowing in presence of a coplanar magnetic vortex.

The experiments were first carried out by E.H. HALL on a thin gold sheet mounted on a glass plate at Johns Hopkins University (Hall, 1879), under the guidance of H.E. ROWLAND.

The motivation for the experiment adduced in HALL's paper is a reasoning on a statement in (Clerk-Maxwell, 1873, vol.II p.144).

The effect is commonly explained in terms of the LORENTZ force, but should be properly reinterpreted on the basis of formula (67) exposed in Prop.6.1 which differs by a factor one-half.

6.7 MUTUAL FORCES BETWEEN PARALLEL ELECTRIC CURRENTS

Let two parallel conducting wires carry electric currents $\mathbf{i}_1$ and $\mathbf{i}_2$. By AMPÈRE law, the integral of the magnetic field $\mathbf{H}$ around circles centered at each current carrying wire is equal to the intensity of the respective electric current. It follows that the intensity of the magnetic field $\mathbf{H}$ and of the corresponding magnetic induction $\mathbf{B}$ decrease linearly with the distance d from the wire, according to the BIOT-SAVART law

$$\mathbf{B} = \mu_0 \mathbf{i}_1 / (2\pi d) . \tag{72}$$

The magnetic induction $\mathbf{B}$ due to $\mathbf{i}_1$ acts on $\mathbf{i}_2$ perpendicularly to the plane of wires and by Eq. (63) generates an electric field $\frac{1}{2}\mathbf{v} \times \mathbf{B}$, with $\mathbf{i}_2 = q\mathbf{v}$, and hence a force per unit length, acting orthogonal to the wires in their plane, proportional to the ratio between the product of currents and their mutual distance, according to the law

$$\mathbf{f} = \tfrac{1}{2} \mu_0 (\mathbf{i}_1 \mathbf{i}_2) / (2\pi d) , \tag{73}$$

which is one-half the standard AMPÈRE force law (Jackson, 1999, (5.11) p.178).

The same for the $\mathbf{B}$ due to $\mathbf{i}_2$ and acting on $\mathbf{i}_1$. The effect of the electric fields will be mutually attractive if the currents flow in the same direction, repulsive otherwise.

6.8 THE RAILGUN: A WEAPON APPLICATION

Let two parallel conductive rails and a sliding or rolling conductive projectile be subject to a high intensity electric current.

The magnetic field generated by the electric current according to the law Eq. $(53)_2$, gives rise to a magnetic vortex field $\omega^2_{\mathbf{B}}$ which acts back on the electric charges in motion along the conductive path, according to the law (63). This last action pushes away one from the other the two rails, which should then be properly fixed to remain in place, and pushes forward the sliding projectile, which undergoes a huge acceleration.

7 CONCLUSIONS

The assumption made on the basis of Eq. (42) that a magnetic induction field $\mathbf{B}$ generates on a moving charge a force that depends only on the values of the field $\mathbf{B}$ and of the spatial velocity field $\mathbf{v}$ just at the position occupied by the charge, is certainly appealing for the electrical engineer which gets therefrom a simple rule to apply for his computations.

The satisfaction is however bound to turn soon into embarrassment when he/she is called to provide a rationale for evaluating the charge velocity.

This notwithstanding the mysterious formula $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ is still suggested without comments in almost every book on electromagnetics.

In the extreme case a still more mysterious argument based on space-time LORENTZ transformations is adduced to shut up questions and comments arising in a natural way from physically minded people. This dogmatic situation is still going on, as a heavy theoretical heritage after more than one century of brilliant technological evolution.

The author spent a large part of the last couple of years in collecting documentation concerning the complex state of affairs and only after taking the bold decision of revisiting the relativistic treatment of electromagnetics he arrived at the conclusion that the whole story did take the wrong path long ago (Romano G., 2013).

After having rewritten the basic laws in terms of differential forms, the evaluation of the derivatives of integrals over moving manifolds can be correctly evaluated and localized in terms of LIE derivatives of the inducting fields along the motion.

The naturality properties of LIE derivatives and of exterior derivatives, assure frame invariance of the rules of electromagnets induction.

On this basis it has been shown that a new, frame invariant theory of classical electromagnetism can be formulated in terms of balance principles involving line integrals and their time derivatives along the motion.

For simultaneity preserving frame transformations, such as the ones considered in the classical (i.e. non-relativistic) theory, invariance of the classical laws of electromagnetic induction is assured.

For frame transformations that do not transform simultaneous events into events still evaluated to be simultaneous by the same observer, as occurs for LORENTZ transformations of special relativity, an entanglement of electric and magnetic fields is detected by an observer who tries to describe the transformed fields from his own point of view.

This entanglement, and the amplification according to the relativistic factor, tend however to vanish in the classical limit and are then negligible when ordinary velocities are involved (Romano G., 2013), a result well expected on physical grounds.

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