Chain Rule for planar bi–Sobolev maps

Nota di L. D'Onofrio, L. Migliaccio, C. Sbordone and R. Schiattarella
Presentata dal socio Carlo Sbordone

(Adunanza del 7 febbraio 2014)

Abstract - For a planar bi–Sobolev map f the zero set of the Jacobian may have positive measure while the regular set does have positive measure. We discuss the compatibility of these facts with the validity of the chain rule:

$$(\star) \qquad J_{f^{-1}}(f(x))\, J_f(x) = 1.$$

It turns out that the left hand side of $(\star)$ is uniquely defined a.e. as a Borel function. This is no more true for bi–Sobolev maps in $\mathbb{R}^n$ with $n \geq 3$.

Mathematics Subject Classification (2010). 46E35, 26B05.

Keywords. bi-Sobolev maps, Jacobian.

1 Introduction

Let Ω, Ω' be domains in $\mathbb{R}^2$. In the category of $W^{1,p}$– Sobolev maps $f : \Omega \overset{\text{onto}}{\longrightarrow} \Omega'$, $f \in W^{1,p}(\Omega, \mathbb{R}^2)$, $p \geq 1$, the case $p = 2$ due to its borderline character exhibits peculiar features, which are considered natural for many reasons. Beyond that natural setting and especially when $p = 1$, some fundamental properties fail and it is rather hard to recover the basic facts which actually require new proofs and methods far from routine. The discrepancy between the $W^{1,2}$ and the $W^{1,1}$ theories is clarified here in case of homeomorphisms.

The Sobolev class of homeomorphisms $f = (u, v) \in W^{1,2}(\Omega, \mathbb{R}^2)$ enjoys some important properties as:

- integration by parts against the Jacobian determinant $J_f(x)$, i.e.

$$\int_\Omega \varphi(u_x v_y - u_y v_x) = \int_\Omega u(\varphi_y v_x - v_y \varphi_x) = \int_\Omega v(\varphi_x u_y - \varphi_y u_x)$$

  for any test function $\varphi \in C_0^\infty(\Omega)$;

- the $(\mathcal{N})$–property of Lusin, that is f maps sets of measure zero to sets of measure zero, briefly $f \in (\mathcal{N})$;

- the usual chain rule for the composition

$$J_f(f^{-1}(y))\,J_{f^{-1}}(y) = 1$$

if f^{-1} is differentiable at y and f is differentiable at $f^{-1}(y)$.

On the other hand, the differentiability almost everywhere holds already for planar homeomorphisms in the Sobolev class $W^{1,1}$ (see [24]). This is not true for $n \geq 3$; indeed, one can construct a nowhere differentiable homeomorphism in $W^{1,n-1}(\Omega, \mathbb{R}^n)$ (see [5]). Here, differentiability is understood in the classical sense.

Relevant work with regularity assumption below $W^{1,2}$ was done by Ziemer [28] who assumed $(\mathcal{N})$ condition for f and f^{-1}. More recently such assumptions have been removed and many interesting results have been found by Hencl and Koskela [18] (see also [19], [5], [21], [12]).

Each homeomorphism f in the Sobolev space $W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ satisfies either $J_f \geq 0$ a.e. or $J_f \leq 0$ (see [20], [1]). We shall assume that $J_f \geq 0$ a.e. in Ω.

For a homeomorphism $f \in W^{1,1}(\Omega, \mathbb{R}^2)$, we decompose the domain Ω as follows:

$$\Omega = \mathcal{R}_f \cup \mathcal{Z}_f \cup \mathcal{E}_f,$$

where

$$\mathcal{R}_f = \{x \in \Omega : f \text{ is differentiable at } x \text{ and } J_f(x) \neq 0\}$$

is the set (possibly empty) of regular points of f,

$$\mathcal{Z}_f = \{x \in \Omega : f \text{ is differentiable at } x \text{ and } J_f(x) = 0\}$$

is the set of singular points of f and

$$\mathcal{E}_f = \{x \in \Omega : f \text{ is not differentiable at } x\}\,.$$

Since f is continuous, these are Borel sets.

In [21] a particularly useful class of homeomorphisms which lie between homeomorphisms of bounded variation and bi–Lipschitz mappings was introduced, namely the *bi–Sobolev mappings* (see [17] for a rich and updated account of the theory which relates these maps with those of finite distortion).

Definition 1. The homeomorphism $f : \Omega \subset \mathbb{R}^2 \xrightarrow{\text{onto}} \Omega' \subset \mathbb{R}^2$ is a bi–Sobolev map if $f \in W^{1,1}_{\text{loc}}(\Omega, \mathbb{R}^2)$ and $f^{-1} \in W^{1,1}_{\text{loc}}(\Omega', \mathbb{R}^2)$.

Here our aim is to study the limits of validity for bi–Sobolev maps of the chain rule for the pointwise Jacobian of the compositions $f \circ f^{-1}$ and $f^{-1} \circ f$, namely

$$J_{f^{-1}}(f(x))J_f(x) = 1 \tag{1.1}$$

and

$$J_f(f^{-1}(y))J_{f^{-1}}(y) = 1 \tag{1.2}$$

Of course, (1.1) is true if $x \in \mathcal{R}_f$. Similarly, (1.2) is valid if f^{-1} is differentiable at y and $J_{f^{-1}}(y) \neq 0$.

It is interesting to note that any bi–Sobolev map in the plane has a set of regular points of positive measure (see Proposition 2.2); this means $|\mathcal{Z}_f| < |\Omega|$.

In the literature, there are some examples of bi–Sobolev mappings such that the Jacobian of f is zero on a set of positive measure (see [26], [23], [22]). How are these examples consistent with (1.1) and (1.2)? The answer to this question is given by the following

Theorem 1.1. *Let $f : \Omega \subset \mathbb{R}^2 \xrightarrow{onto} \Omega' \subset \mathbb{R}^2$ be a bi–Sobolev map; then we can define $J_{f^{-1}}(f(x))J_f(x)$ uniquely as a measurable function by*

$$J_{f^{-1}}(f(x))J_f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{R}_f \\ \\ 0 & \text{if } x \in \mathcal{Z}_f \end{cases} \tag{1.3}$$

Remark 1.1. We would like to underline that the composition of two maps $h \circ g$, $g : \Omega \to \Omega'$, $h : \Omega' \to \mathbb{R}$ with the outer map h defined a.e., may be undefined on a set of positive measure $P \subset \Omega$. Indeed, $J_{f^{-1}} \circ f$ is sensitive to changes of $J_{f^{-1}}$ on subsets of Ω with zero Lebesgue measure. This poses some problems for the definition and the measurability of the composition $J_{f^{-1}} \circ f$.

Remark 1.2. Of course analogous formula is valid for $J_f(f^{-1}(y))J_{f^{-1}}(y)$ for a.e. $y \in \Omega'$.

2 Preliminaries

By Gehring–Lehto theorem, a planar homeomorphism $f \in W^{1,1}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is differentiable a.e. (see [11]). Hence, using the notation introduced above:

$$|\mathcal{E}_f| = 0.$$

If $x \in \mathcal{R}_f$, then f^{-1} is differentiable at $f(x)$ and moreover we have

$$f(\mathcal{R}_f) = \mathcal{R}_{f^{-1}} \tag{2.1}$$

If $f \in W^{1,1}_{\mathrm{loc}}\left(\Omega, \mathbb{R}^2\right)$ is a homeomorphism and η is a nonnegative Borel measurable function on $\mathbb{R}^2$, the following *area inequality* holds

$$\int_B \eta\left(f(x)\right) J_f(x)\, dx \leq \int_{f(B)} \eta(y)\, dy \tag{2.2}$$

for every $B \subset \Omega$ Borel set ([9, Theorem 3.1.8]).

However, the $(\mathcal{N})$ property of Lusin is equivalent to the *area formula*, that is,

$$\int_B \eta\left(f(x)\right) J_f(x)\, dx = \int_{f(B)} \eta(y)\, dy\,, \tag{2.3}$$

for all such functions η.

It is well known (see [21], for example) that there exists a set $\widetilde{\Omega} \subset \Omega$ of full measure such that the area formula always holds for f on $\widetilde{\Omega}$. Notice that the area formula holds on the set $\mathcal{R}_f \cup \mathcal{Z}_f$ where f is differentiable ([17, Corollary A.2]). As a consequence, we have the equality

$$\int_{\mathcal{Z}_f} J_f \, dx = |f(\mathcal{Z}_f)|$$

which implies the following version of the Sard's Lemma

$$|f(\mathcal{Z}_f)| = 0 \,. \tag{2.4}$$

Proposition 2.1. *Let $f : \Omega \subset \mathbb{R}^2 \overset{onto}{\longrightarrow} \Omega' \subset \mathbb{R}^2$ be a bi–Sobolev map. Then*

$$J_f = 0 \ \ a.e \qquad \Longleftrightarrow \qquad J_{f^{-1}} = 0 \ \ a.e. \tag{2.5}$$

Proof. Indeed, by area formula for Sobolev homeomorphisms, there exists $N_0 \subset \Omega$ with zero measure such that

$$\int_{\Omega \setminus N_0} J_f \, dx = |f(\Omega \setminus N_0)|.$$

Since $J_f = 0$ a.e. then

$$0 = |f(\Omega \setminus N_0)| = |\Omega' \setminus f(N_0)|$$

and

$$\int_{f(N_0)} J_{f^{-1}} \, dy \leq |N_0| = 0$$

It implies that $J_{f^{-1}} = 0$ a.e. By symmetry we obtain the other implication. $\square$

The next Proposition was proved in [16] using the language of mappings of finite distortion. The same result with a different proof was proved in [6].

Theorem 2.1. *If $f = (u, v)$ is a $W^{1,1}(\Omega, \mathbb{R}^2)$-homeomorphism, then $f^{-1} = (s, t) \in BV(\Omega', \mathbb{R}^2)$ and*

$$|\nabla s|(\Omega') = \int_\Omega \left| \frac{\partial f}{\partial t} \right| \mathrm{d}z$$

$$|\nabla t|(\Omega') = \int_\Omega \left| \frac{\partial f}{\partial s} \right| \mathrm{d}z.$$

Here $|\nabla s|(\Omega')$ and $|\nabla t|(\Omega')$ are the total variations of the coordinate functions of the inverse f^{-1}.

For the sake of completeness we present here the proof of the following

Proposition 2.2. *If f is a planar bi–Sobolev map, then $0 < |\mathcal{R}_f|$.*

Proof. We prove that $|\mathcal{R}_f| > 0$ by showing that $|\mathcal{Z}_f| < |\Omega|$. Suppose by contradiction that the pathological equality

$$|\mathcal{Z}_f| = |\Omega|$$

holds. By (2.5), $|\mathcal{Z}_{f^{-1}}| = |\Omega'|$ and hence there exists $N_0' \subset \Omega' : |N_0'| = 0$ such that

$$\int_{\Omega' \setminus N_0'} J_{f^{-1}} \, dy = |\Omega \setminus f^{-1}(N_0')| = 0$$

It means that f^{-1} sends the null set N_0' to a set $f^{-1}(N_0')$ of full measure. By regularity properties of Lebesgue measure there exists a closed set $C \subset f^{-1}(N_0')$ such that $|C| > 0$. Define $C' = f(C)$ and notice that it is a closed subset of N_0' with zero measure: $|C'| = 0$. Denoting by $f = (u, v)$ and $f^{-1} = (s, t)$, we apply Theorem 2.1 to the open set $A' = \Omega' \setminus C'$, recalling that, as the inverse belongs to $W^{1,1}$ by assumption the variation coincide with the integral of the gradient of the coordinate functions, to obtain:

$$\int_{A'} |\nabla s| = \int_{f^{-1}(A')} \left| \frac{\partial f}{\partial t} \right|$$

$$\int_{A'} |\nabla t| = \int_{f^{-1}(A')} \left| \frac{\partial f}{\partial s} \right|$$

Therefore, we get

$$\begin{aligned} \int_{C'} |\nabla s| &= \int_{\Omega'} |\nabla s| - \int_{A'} |\nabla s| \\ &= \int_{f^{-1}(\Omega')} \left| \frac{\partial f}{\partial t} \right| - \int_{f^{-1}(A')} \left| \frac{\partial f}{\partial t} \right| = \int_{f^{-1}(C')} \left| \frac{\partial f}{\partial t} \right| > 0 \end{aligned}$$

because $|C'| = 0$ and $|f^{-1}(C')| > 0$. This means that ∇s is not absolutely continuous with respect to Lebesgue measure and $f^{-1} \notin W^{1,1}_{loc}$ which gives the contradiction.

$\square$

Remark 2.1. Proposition 2.2 is false in dimension $n = 3$ as derives by a recent example of a bi-Sobolev map f such that $|\mathcal{R}_f| = |\mathcal{R}_{f^{-1}}| = 0$ (see [7]).

3 Chain Rule for the pointwise Jacobian

Proposition 3.1. *Let $f : \Omega \xrightarrow{onto} \Omega'$ be a planar bi–Sobolev map. Then the following statements are equivalent:*

1) *f^{-1} does not satisfy the Lusin condition*

2) $|\mathcal{Z}_f| > 0$

3) $|f^{-1}(\mathcal{E}_{f^{-1}})| > 0$

Proof. 1) $\Longrightarrow$ 2) If f^{-1} does not satisfy $(\mathcal{N})$ condition, there exists a Borel set $A \subset \Omega$ such that $|A| > 0$ and $|f(A)| = 0$. By (2.2)

$$\int_A J_f \, dx \leq |f(A)| = 0$$

and therefore $J_f = 0$ for almost every $x \in A$. Hence,

$$|A \setminus \mathcal{Z}_f| = 0.$$

2) $\Longrightarrow$ 3) By (2.1) we have

$$f^{-1}(\mathcal{Z}_{f^{-1}}) \cup f^{-1}(\mathcal{E}_{f^{-1}}) = f^{-1}\left(\mathcal{Z}_{f^{-1}} \cup \mathcal{E}_{f^{-1}}\right) = \mathcal{Z}_f \cup \mathcal{E}_f. \tag{3.1}$$

Hence, essentially, f^{-1} maps $\mathcal{E}_{f^{-1}}$ to $\mathcal{Z}_f$ in the sense that, as (3.1) yields

$$f^{-1}(\mathcal{E}_{f^{-1}}) \triangle \mathcal{Z}_f = \left(f^{-1}(\mathcal{E}_{f^{-1}}) \setminus \mathcal{Z}_f\right) \cup \left(\mathcal{Z}_f \setminus f^{-1}(\mathcal{E}_{f^{-1}})\right) \subset \mathcal{E}_f \cup f^{-1}(\mathcal{Z}_{f^{-1}})$$

and we have

$$|f^{-1}(\mathcal{E}_{f^{-1}}) \triangle \mathcal{Z}_f| = 0 \tag{3.2}$$

that implies

$$|f^{-1}(\mathcal{E}_{f^{-1}})| = |\mathcal{Z}_f|$$

3) $\Longrightarrow$ 1) As f^{-1} is differentiable almost everywhere, then $|\mathcal{E}_{f^{-1}}| = 0$. Hence, f^{-1} sends the null set $\mathcal{E}_{f^{-1}}$ onto a set of positive measure and this means that f^{-1} does not satisfy Lusin condition. $\square$

Now we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. It is irrelevant to define $J_{f^{-1}}(f(x))J_f(x)$ for $x \in \mathcal{E}_f$ because bi-Sobolev map in the plane satisfies $|\mathcal{E}_f| = 0$ by Gehring-Lehto Theorem.

For all $x \in \mathcal{R}_f$, the differential matrices of f and f^{-1} are inverse to each other, namely:

$$Df^{-1}(f(x)) = (Df(x))^{-1} \qquad \forall x \in \mathcal{R}_f \tag{3.3}$$

and

$$J_{f^{-1}}(f(x))J_f(x) = 1 \qquad \forall x \in \mathcal{R}_f. \tag{3.4}$$

It remains to study $J_{f^{-1}}(f(x))J_f(x)$ for $x \in \mathcal{Z}_f$.

If $|\mathcal{Z}_f| > 0$, the function $J_{f^{-1}} \circ f$ a priori is not well defined. Indeed, by Sard's Lemma $|f(\mathcal{Z}_f)| = 0$ and we can choose arbitrarily $J_{f^{-1}}$ on $f(\mathcal{Z}_f)$, that is $J_{f^{-1}}(f(x))$ for x belonging to the positive set $\mathcal{Z}_f$ and the composed function $J_{f^{-1}} \circ f$ may be non measurable on the set of positive measure $\mathcal{Z}_f$. In fact as x varies in $\mathcal{Z}_f$, we are dealing with points $f(x)$ filling a zero subset of Ω' and so we can redefine $J_{f^{-1}}$ arbitrarily on such zero subset. However, the product

$$J_{f^{-1}}(f(x))\, J_f(x)$$

is zero as $J_f(x) = 0$ for $x \in \mathcal{Z}_f$.

On the other hand, if $|\mathcal{Z}_f| = 0$, by Proposition 3.1, f^{-1} satisfies Lusin $(\mathcal{N})$ condition and the image of a null set is a null set. Therefore, the composition $J_{f^{-1}} \circ f$ has a meaning as a measurable function. As $|\mathcal{Z}_f| = 0$ then $|\mathcal{R}_f| = |\Omega|$ and by (3.4) follows that

$$J_{f^{-1}}(f(x))\, J_f(x) = 1 \qquad \text{a.e. } x \in \Omega. \tag{3.5}$$

The definition (1.3) is of course the unique possible to get a measurable function.

□

Remark 3.1. Formula (1.3) may be ambiguous for $n = 3$; the reason is that there exists a bi–Sobolev map f that is nowhere differentiable together with its inverse ([5]), and so $|\mathcal{E}_f| = |\Omega|$. Hence one can define $\big(J_{f^{-1}} \circ f\big)\, J_f = \chi_{\mathcal{R}_f \cup \mathcal{E}_f}$ or $\big(J_{f^{-1}} \circ f\big)\, J_f = \chi_{\mathcal{R}_f}$ and these two functions disagree almost everywhere.

Acknowledgement: We thank prof. J.M.Ball for the stimulating questions he addressed to the third and the fourth author, after their Lectures at the Conference "Vector-valued Partial Differential Equations and Applications CIME-EMS Summer School"dedicated to prof. B. Dacorogna in occasion of his 60th birthday, Cetraro, It, 8–12, 2013.

References

[1] K. Astala, T. Iwaniec, G. Martin. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series vol. 24, (2008).

[2] J. M. Ball, *Global invertibility of Sobolev functions and the interpenetration of matter*, Proc. Roy. Soc. Edinburgh Sect. A 88 (1981), no. 3-4, 315–328.

[3] B. Bojarski, T. Iwaniec: *Analytical foundations of the theory of quasiconformal mappings in* $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 8, 2, (1983), 257-324.

[4] R. Cerný, *Homeomorphism with zero Jacobian: Sharp integrability of the derivative*, J. Math. Anal. Appl, 373, (2011), 161–174.

[5] M. Csörnyei, S. Hencl, J. Malý: *Homeomorphisms in the Sobolev space* $W^{1,n-1}$. J. Reine Angew. Math., 644, (2010), 221-235.

[6] P. Di Gironimo, L. D'Onofrio, C. Sbordone, R. Schiattarella: *Anisotropic Sobolev homeomorphisms*, Ann. Acad. Sci. Fenn. Math., 36, (2011), 593–602.

[7] L. D'Onofrio, S. Hencl and R. Schiattarella, *Bi-Sobolev homeomorphism with zero Jacobian almost everywhere*, Calc. Var., DOI 10.1007/s00526-013-0669-6.

[8] L. D'Onofrio, C. Sbordone and R. Schiattarella, *The Grand Sobolev homeomorphisms and their measurability properties*, Advanced Nonlinear Studies, 12 (2012), no. 4, 767–782.

[9] H. Federer: *Geometric Measure Theory*, Die Grundlehren der mathematischen Wissenschaften, Band 153 Springer - Verlag, New York (1969) (Second edition 1996).

[10] N. Fusco, G. Moscariello, C. Sbordone: *The limit of $W^{1,1}$-homeomorphisms with finite distortion*, Calc. Var. , 33, (2008), 377-390.

[11] F. W. Gehring and O. Lehto: *On the total differentiability of functions of a complex variable*, Ann. Acad. Sci. Fenn. Ser. A I 272, (1959), 1–9.

[12] L. Greco, C. Sbordone and C. Trombetti, *A note on planar homeomorphisms*, Rend. Accad. Sci. Fis. Mat. Napoli (4) 75 (2008), 53-59.

[13] P. Hajlasz, *Change of variables formula under minimal assumptions*, Coll. Math. 64 93–101.

[14] J. Heinonen, T. Kilpeläinen and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Dover Publications, Inc., Mineola, NY, 2006.

[15] S. Hencl: *Sharpness of the assumptions for the regularity of a homeomorphism,* Michigan Math. J. 59 (2010), no. 3, 667–678.

[16] S. Hencl: *Sobolev homeomorphism with zero Jacobian almost everywhere*, J. Math. Pures Appl. (9) 95 (2011), no. 4, 444-458.

[17] S. Hencl and P. Koskela: *Lecture notes on Mappings of finite distortion*, Lecture Notes in Mathematics, Springer, vol. 2096, (2014).

[18] S. Hencl, P. Koskela: *Regularity of the inverse of a planar Sobolev homeomorphism*, Arch. Ration. Mech. Anal., 180, (2006), 75-95.

[19] S. Hencl, P. Koskela and J. Malý: *Regularity of the inverse of a Sobolev homeomorphism in space*, Proc. Roy. Soc. Edinburgh Sect. A, 136:6, (2006), 1267-1285.

[20] S. Hencl, J. Malý: *Jacobians of Sobolev homeomorphisms*, Calc. Var. Partial Differential Equations 38 (2010), no. 1–2, 233–242.

[21] S. Hencl, G. Moscariello, A. Passarelli di Napoli and C. Sbordone: *Bi-Sobolev mappings and elliptic equations in the plane*, J. Math. Anal. Appl., 355, (2009), 22-32.

[22] T. Iwaniec and G. Martin, *Geometric function theory and non-linear analysis*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2001.

[23] T. Iwaniec and G. Martin, *Squeezing the Sierpin'ski sponge*, Studia Math. 149 (2002), no. 2, 133-145.

[24] O. Lehto, K. Virtanen: *Quasiconformal Mappings in the Plane*, Springer- Verlag, Berlin, (1971).

[25] O. Martio, S. Rickman, J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 448 (1969), 1–40.

[26] S. P. Ponomarev, *Property N of homeomorphism in the class* $W^{1,p}$, Transl. Sibirskii Math., 28, (2) (1987), 140–148.

[27] Yu. G. Reshetnjak, *Spatial mappings with bounded distortion.* Sibirsk. Mat. Z. 8, (1967) 629-658.

[28] W.P. Ziemer: *Change of variables for absolutely continuous functions*, Duke Math. J. ,36:1, (1969), 171–178.

Luigi D'Onofrio
Dipartimento di Statistica e Matematica per la Ricerca Economica
Università "Parthenope", Via Parisi 13, 80100 Napoli, Italy

e-mail: donofrio@uniparthenope.it

Lucia Migliaccio & Carlo Sbordone & Roberta Schiattarella
Università di Napoli "Federico II", Via Cintia, 80126 Napoli, Italy

e-mail:lucia.migliaccio@unina.it
e-mail: sbordone@unina.it
e-mail: roberta.schiattarella@unina.it